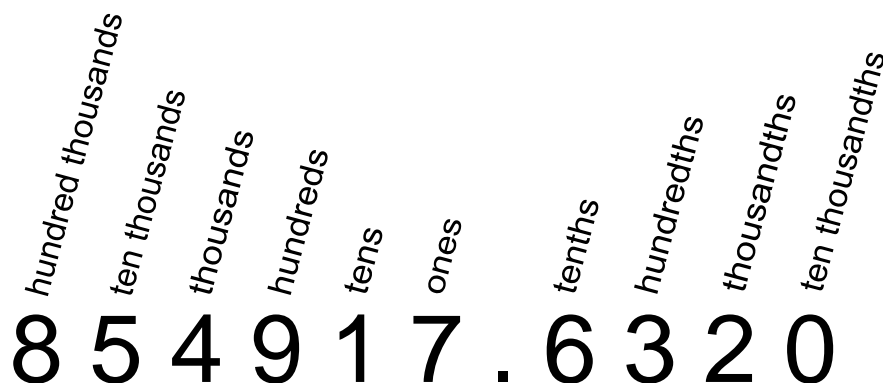# Converting Between Bases

Through most of the world today we use the **decimal** or **base 10 system** of numbers. The "10" there tells us two things: our numbering system has ten different digits in it, and each **place value** in the expression of a number as a numeral is ten times the previous one. You might remember this diagram from grade school:

We can write our ten digits in any combination, and it's meaningful because the position of each digit tells us its value. (Compare this with something like Roman numerals where 1, 10, 100 and 1000 all need different symbols, and not every combination of those symbols is meaningful — LV is a Roman numeral; LVL is not.)

Let's consider integers only. So why does a number like 425 mean what it does? The last digit, the 5, is always in the ones position. In this number, that's because $10^0 = 1$. Then the number next to that, the 2, is in the tens place because $10^1 = 10$. The pattern continues: the 4 is in the hundreds place because $10^2 = 100$. If there were more digits, they would take values of 10^3, 10^4, 10^5, and so on. The value of "425" is:

$(4 \times 10^2) + (2 \times 10^1) + (5 \times 10^0)$.

The choice of 10 for our base was an arbitrary one, made solely because the typical human has ten fingers. Mathematically speaking, it's not a great choice. Division and multiplication are harder than they need to be, and it makes most fractions harder to work with than, say, a number system based on 12s would be.

We can express any integer in any base. It's just a matter of converting from one base to the other. There's one technique for going from other bases to decimal; one for going from decimal to other bases, and a special technique for conversions between bases useful in computer science.

Authored by Darren Rigby

BASE CONVERSION: THE GENERAL CASE

To convert a number from some other base to decimal is easy. We just need to calculate the value of each digit and add all the digits up.

Example 1:   Express the number 425_7 in base 10.

Solution: The subscript "7" tells us that this numeral is written in base 7. The system only uses the digits 0, 1, 2, 3, 4, 5 and 6 (only seven digits) and has a place-value system based on powers of 7. The 5 is in the ones place — always — then the 2 is in the "sevens" place, since $7^1 = 7$, and the 4 is in the "forty-nines" place, since $7^2 = 49$. The value of 425_7 is $(4 \times 7^2) + (2 \times 7^1) + 5 = 196 + 14 + 5 = 215_{10}$.

Converting in the other direction is a division process, rather than a multiplication, and involves "peeling off" remainders.

Example 2:   Express the number 4625_{10} in base 12.

Solution: We would expect the answer might be a four-digit answer. (If it's not, it's not important to the logic that's about to follow.) Then the breakdown of that four-digit answer in base 12 would look like this:

$$(w \cdot 12^3) + (x \cdot 12^2) + (y \cdot 12^1) + z$$

We can factor 12's out of the expression to get:

$$\{[(w) \cdot 12 + x] \cdot 12 + y\} \times 12 + z$$

Since we want z to be a digit in base 12, z must be less than 12. We can think of it, then as the remainder that results when 4625_{10} is divided by 12, and $\{[(w) \cdot 12 + x] \cdot 12 + y\}$ is the quotient in that division. $4625 \div 12 = 385$ R 5. Then $385 = [(w) \cdot 12 + x] \cdot 12 + y$, and the y is the remainder when 385 is divided by 12. We can keep going, dividing by 12 and taking away remainders until we get a result that is less than 12. This process gives us the digits in the base 12 expression for 4625_{10} from last to first. We find that the remainders (in reverse order) are 2815, so $4625_{10} = 2815_{12}$.

There are calculators that can perform division with remainders, but many don't. You can still do these problems on a calculator relatively easily. Type the division problem into the calculator as normal, and you'll get an answer that's a decimal fraction. Write down the whole number part of the answer, and subtract that whole number in the calculator; you should get just the fraction part of the answer. Multiply this fraction by the base to get the remainder:

4	6	2	5	÷	1	2	=			385.41667
−	3	8	5	=						0.4166667
×	1	2	=							5.

What are the digits in base 12? In bases higher than 10, we start using letters as the extra digits. In base 12, they're 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A and B. You're not likely to be asked to work in a base above base 16 (…, C, D, E and F), and that's because…

BASE CONVERSION FOR COMPUTER SCIENCE: THE SPECIAL CASE

Computer scientists and coders frequently work in base 2 (**binary**), base 8 (**octal**) and base 16 (**hexadecimal**, or just **hex**). Computers naturally work in binary code, and the special case of this section of this worksheet allows us to take advantage of octal and hex to make calculation and communication of binary numbers easier among people.

The binary system has just two digits, 0 and 1. Converting to and from binary can be a bit faster if you know the first eight powers of two: 1, 2, 4, 8, 16, 32, 64, 128.

Example 3: a) Express the number 54_{10} in base 2.
b) Express the number 1010110_2 in base 10.

Solution: Since no digit higher than 1 can be involved in these questions, we're not multiplying or dividing, just adding and subtracting.
a) The largest power of two less than 54 is 32. There's a 1 in the 32s place, and
$54 - 32 = 22$. Now start counting down the powers of two, starting with the 32 we just looked at. It's a 1 if it can be subtracted from the new total; it's a 0 if it can't.
$22 - 16 = 6$; 1 in the 16s place
6—too small; 0 in the 8s place
$6 - 4 = 2$; 1 in the 4s place
$2 - 2 = 0$; 1 in the 2s place
0—too small; 0 in the 1s place. $\therefore 54_{10} = 110110_2$
b) Add up all the places that contain a 1:

1	0	1	0	1	1	0
64	32	16	8	4	2	1

$64 + 16 + 4 + 2 = 86$. $\therefore 1010110_2 = 86_{10}$.

Rather than saying or writing a long string of 0s and 1s, coders and computer scientists will commonly "bundle" bits together and express them in hex. We often see these in HTML colors and Unicode codepoints.

Example 3: Express the number 1011010_2 in base 16.

Solution: Bundle the digits together into groups of 4, starting at the ones place and working left. Then translate each bundle into a hex digit all by itself.

$$101 \; 1010$$
$$5 \quad 10$$

The digit "10" comes after "9", so instead we use the hex digit A. $\therefore 1011010_2 = 5A_{16}$

EXERCISES
A. Because $3^3 = 27$, it's handy to encode English letters of the alphabet and the space in base 3 (ternary). Decrypt the following message written in three-digit ternary numbers: Spaces are 0 in decimal; letters are encoded by their position in the alphabet.
202 022 012 000 002 012 201 202 000 202 022 100 112 021 201 000 100 112 000
120 112 020 012 000 001 200 012 000 202 022 200 022 022

B. The number 12345_{10} is equal to either 5064_7 or 50664_7. Which is it? Justify your choice. [*Hint:* Do you expect your answer to look larger or smaller in this base?]

C. Express the following in base 10.
   1) 3132_4
   2) 831_9
   3) 1042_5
   4) $1A7_{11}$

D. Express the number 100_{10} in the following in bases.
   1) base 3
   2) base 6
   3) base 13
   4) base 8

E. Express the following numbers in the given bases by converting to decimal first.
   1) 3456_7 into base 12
   3) $3B_{14}$ into base 2

   2) 33201_4 into base 9
   4) 10005_6 into base 15

F. Angles expressed in degrees, minutes and seconds can be thought of as being expressed in base 60. Convert these angles into D°M′S″ format using the calculator technique from page 2.
   1) 37.94725°
   2) 121.74409°
   3) 3.18902°
   4) 18.022759°

G. Convert these bytes into two hexadecimal nybbles.
   1) 11010010
   3) 01111100
   5) 01101000
   7) 00110001

   2) 00100110
   4) 10011010
   6) 11100100
   8) 10111111

H. Convert these numbers into a byte of binary digits and a hex value.
   1) 17
   3) 59
   5) 89
   7) 156

   2) 38
   4) 74
   6) 127
   8) 214

SOLUTIONS

A: THE BEST THINGS IN LIFE ARE THREE

B: It should be 50664_7. Since base 7 has fewer digits, we should expect numbers to look larger *faster* as we count. (Consider binary expansions — to express numbers up to 255_{10}, a three-digit decimal number, takes up to eight binary digits.) Since "5064" looks smaller than the number we started with, it can't be the base-7 expression of the same number.

C: (1) $(3 \cdot 64) + (1 \cdot 16) + (3 \cdot 4) + 2 = 222_{10}$ (2) $(8 \cdot 81) + (3 \cdot 9) + 1 = 676_{10}$
(3) $(1 \cdot 125) + (0 \cdot 25) + (4 \cdot 5) + 2 = 147_{10}$ (4) $(1 \cdot 121) + (10 \cdot 11) + 7 = 238_{10}$

D: (1) 10201_3 (2) 244_6 (3) 79_{13} (4) 144_8

E: (1) $1266_{10} = 896_{12}$ (2) $993_{10} = 1323_9$ (3) $53_{10} = 110101_2$ (4) $1301_{10} = 5BB_{15}$

F: (1) 37°56′50.1   (2) 121°44′38.724″   (3) 3°11′20.472″   (4) 18°0′9.9324″

G: (1) D2   (2) 26   (3) 7C   (4) 9A   (5) 68   (6) E4   (7) 31   (8) BF

H: (1) 00010001, 11   (2) 00100110, 26   (3) 00111011, 3B   (4) 01001010, 4A
(5) 01011001, 59   (6) 01111111, 7F   (7) 10011100, 9C   (8) 11010110, D6