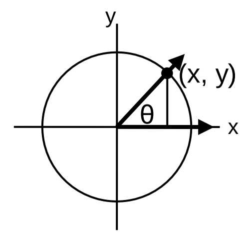
Essentials of Trigonometry 2:
Beyond Right Triangles

THE UNIT CIRCLE

Trigonometry with right triangles can only tell us about acute angles, but we can extend the concept of trig ratios to obtuse angles, **reflex angles** (angles measuring between 180° and 360°), and even angles measuring greater than 360° or less than 0°. To do this we create right triangles within a **unit circle**, a circle drawn with its centre at (0, 0) and a radius of 1.



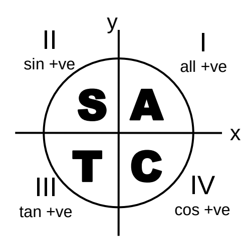
In using the unit circle, the angle θ starts at the positive x-axis, and opens counterclockwise. The arm of the angle that isn't on the x-axis intersects the unit circle at a point (x, y). We create a triangle by drawing a vertical line between (x, y) and the x-axis. If θ is an acute angle, then the six trigonometric ratios can be calculated using those coordinates and the radius of the circle, r.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{x}{r} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x}$$

$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{r}{y} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{r}{x} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{x}{y}$$

Because of the specific way the triangle is constructed, hypotenuse always goes with r, adjacent goes with x, and opposite goes with y.

For acute angles, x and y are always positive. We can form a trigonometric ratio for angles that are 90° or more by going past the positive y-axis on the way around the circle. In these cases, the x- or y-coordinate (or both) might be negative; r is never negative because it is considered to be a distance, and distances cannot be negative.



We divide the unit circle into **quadrants**. They're numbered as shown in the diagram. The values of each ratio are consistently positive or negative within a quadrant, but change from quadrant to quadrant. Remember that if a particular ratio is positive (or negative), then its reciprocal will have the same sign. In this diagram, then, only the three main ratios are shown. In quadrant I everything is positive. In quadrant II, sin is positive; cos and tan are negative. In quadrant III, tan is positive; sin and cos are negative. In quadrant IV, cos is positive; sin and tan are negative. Memorize this is by remembering CAST: positive <u>c</u>os, <u>a</u>ll, <u>s</u>in, <u>t</u>an starting from quadrant IV and moving counterclockwise.

We form trig ratios for angles higher than 360° by going around the unit circle more than once. For negative angles, go clockwise from the positive x-axis rather than counterclockwise. A unit circle diagram for any angle outside the range 0° – 360° looks just like the diagram for some angle between 0° and 360° — they cross the unit circle in

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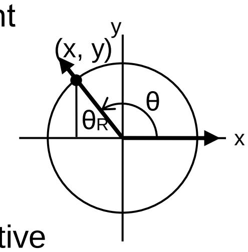
the same place. (The diagrams for 370° and −350°, for example, both look the same as the one for 10°.) We say that angles over 360° and negative angles are **co-terminal** with an angle between 0° and 360° if they cross the unit circle at the same point.

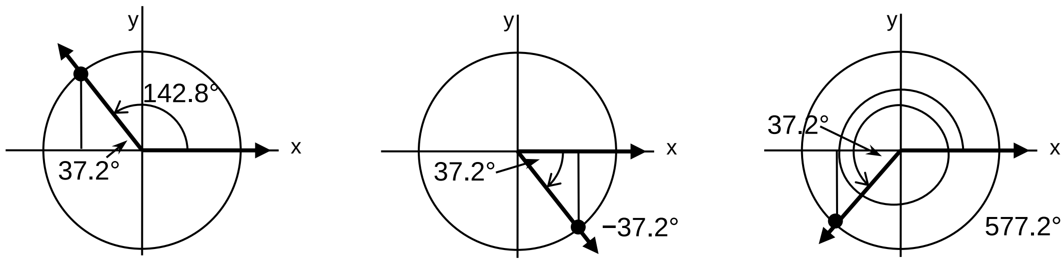


With angles that aren't acute, we follow the same steps for drawing right triangles within a unit circle: add a vertical line from the point where the moving arm of the angle crosses the circle. We then work with the angle inside the triangle. The diagram at right shows how we would draw an obtuse angle θ on the unit circle. We would derive the trig ratios from the triangle containing $θ_R$. The ratios that involve x for this angle are negative, because the x-coordinate of the point (x, y) is negative in this position. The angle that we use to calculate the trigonometric ratios ($θ_R$, in the example) is called a **reference angle**.

Example 2: In the unit circle diagram for 37.2° the arm of the angle crosses the unit circle at the point (0.8, 0.6). Find (a) cos 142.8°, (b) sin −37.2° and (c) tan 577.2°.

Solution: Diagrams will help us see what these angles look like, and what the reference angles will be.

(a) 142.8° (b) −37.2° (c) 577.2°



(a) The angle 142.8° is in quadrant II. To find the reference angle in quadrant II we find the difference between the angle in the question and 180°. 180° − 142.8° = 37.2°. That's the reference angle. cos 37.2° = 0.8 ÷ 1 = 0.8. In quadrant II only sine is positive, so cos 142.8° = −0.8.

(b) The angle −37.2° is negative, so we draw the angle clockwise from the x-axis. The angle is in quadrant IV and 37.2° is the reference angle. sin 37.2° = 0.6 ÷ 1 = 0.6. In quadrant IV only cosine is positive, so sin −37.2° = −0.6.

(c) The angle 577.2° is greater than 360°. We subtract 360° to know what angle is co-terminal with 577.2°. 577.2° − 360° = 217.2°. 217.2° is in quadrant III. Since the angle is in quadrant III, subtract 180° from the given angle to find the reference angle. 217.2° − 180° = 37.2°, so that is our reference angle. tan 37.2° = 0.6 ÷ 0.8 = 0.75. In quadrant III tan is positive, so tan 577.2° = 0.75.

RADIAN MEASURE

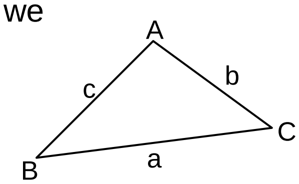
Another unit used for measuring angles is the **radian**. Rather than dividing the circle into 360 sections, we measure the arc length the angle would have on a unit circle. A 360° angle covers the full circle, so the arc length is the whole circumference, or 2π radians (sometimes written "2π rad" or just "2π"). A 180° angle is half that, or π radians. 1 rad ≈ 57.3°, but this usually isn't a useful conversion to know. To convert from degrees to radians, multiply by $\frac{\pi}{180}$. To convert back, divide by $\frac{\pi}{180}$.

You will be expected to know the trigonometry of the angles from the special triangles (30°, 45°, 60°). This includes any angles with reference angles at 30°, 45°, or 60° (and the equivalent angles in radians). Any angle that is based on a 45° angle will be based on $\frac{\pi}{4}$ ("pi by four") radians.

Multiples of 45° angles count by $\frac{\pi}{4}$: 45° = $\frac{\pi}{4}$, 90° = $\frac{2\pi}{4}$ = $\frac{\pi}{2}$, 135° = $\frac{3\pi}{4}$, 180° = $\frac{4\pi}{4}$ = π, and so on. Multiples of 60° angles count by $\frac{\pi}{3}$: 60° = $\frac{\pi}{3}$, 120° = $\frac{2\pi}{3}$, 180° = $\frac{3\pi}{3}$ = π, and so on. Multiples of 30° count by $\frac{\pi}{6}$. To practice more with radians, work on the Learning Centre worksheet "The Unit Circle Explorer's Guide".

SINE LAW & COSINE LAW

Many problems in the real world involve triangles that aren't right triangles. (Any triangle that isn't a right triangle is called an oblique triangle.) In oblique triangles, we can't use the trigonometry of hypotenuse, adjacent, and opposite sides, but we can use the sine law and cosine law.



Recall that the side opposite an angle is the one side in a triangle that is not used in forming the angle. Angles and their opposite sides are labelled with the same letter of the alphabet, as in the diagram.

The **sine law** expresses relationships between angles and their opposite sides:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The sine law still works when all the fractions are flipped (so sine is on the bottom).

Example 3: In the diagram at the top of this page, angle A is 105° and angle B is 25°. If b = 13 cm, how long is side c?

Solution: We can use the last two fractions of the sine law to solve this problem.

$$\frac{\sin B}{b} = \frac{\sin C}{c}$$

To find angle C, we know the three angles add up to 180°, so angle C is 180° − 105° − 25° = 50°.

$$\frac{\sin 25°}{13} = \frac{\sin 50°}{c}$$
$$c \sin 25° = 13 \sin 50°$$
$$c = \frac{13 \sin 50°}{\sin 25°} \approx 23.56 \text{ cm}$$

In order to be able to use the sine law, we must know both an angle and its opposite side. In any other circumstance, we can use the **cosine law**:

$a^2 = b^2 + c^2 - 2bc \cos A \qquad b^2 = a^2 + c^2 - 2ac \cos B \qquad c^2 = a^2 + b^2 - 2ab \cos C$

There are three versions of the cosine law, but they're really all the same. The angle that you know (or are interested in knowing, if you've been given only the sides) is the one you'll be taking the cosine of, and its opposite side is isolated on the other side of the equal sign.

Example 4: In the diagram at the top of the page, a = 12 cm, c = 7 cm and angle B measures 28°. Determine the length of side b.

Solution: We don't have an angle-opposite side pair, so we use cosine law. Since we know angle B, we'll use the second version of the formula.

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cos B \\ &= 12^2 + 7^2 - 2 \cdot 12 \cdot 7 \cos 28° \approx 44.665\ldots \\ \therefore b &= \sqrt{44.665\ldots} \approx 6.68 \text{ cm} \end{aligned}$$

INVERSE TRIGONOMETRIC FUNCTIONS

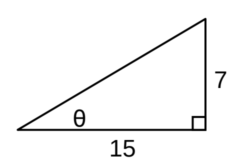
The Essentials of Trigonometry 1 worksheet showed you how to use the angles and one side in a right triangle to determine the lengths of the other sides. What if you knew the sides, but not the angle? You could calculate the trig ratios, but how would you determine the angles that go with the ratios?

The **inverse trigonometric functions** perform that service. You provide the ratio, and the function tells you the angle. An inverse trig function is indicated by writing a superscript "−1" above the usual symbol for the ratio. For example, the **$\sin^{-1}$** ratio ("inverse sine") will tell you the angle if you give it the ratio of the opposite side to the hypotenuse of a right triangle.

Although $\sin^2 x$ means $(\sin x)^2$, the symbol "$\sin^{-1} x$" is <u>*not*</u> the same thing as $(\sin x)^{-1}$! The proper way to write $(\sin x)^{-1}$ is "csc x". The "$^{-1}$" in "$\sin^{-1} x$" isn't an exponent, but a symbol indicating an inverse function, as in f^{-1}. Writing "arc" in front of the function (e.g. "arcsin") means the same thing as an inverse function, but isn't used as often.

You can access the inverse trig functions for sine, cosine and tangent on your calculator by using the [SHIFT] or [2nd Fn] keys with the usual trig function keys.

Example 1: Determine the measure of the angle θ.



Solution: To decide what ratio to use, look at what information is given. We have the opposite and adjacent sides, so we use tan. We can set up an equation to solve for the angle:

$$\begin{aligned} \tan\theta &= \tfrac{\text{opposite}}{\text{adjacent}} = \tfrac{7}{15} = 0.46666\ldots \\ \theta &= \tan^{-1} 0.46666\ldots \\ &= 25.01689\ldots \approx 25° \end{aligned}$$