



Completing the Square

When asked to solve a quadratic equation, the strategy of isolating x doesn't work with both x^2 - and x -terms. We have to factor. The sum-and-product method (or the ac method) is the fastest way, but doesn't work for all quadratics. For example,

$$x^2 + 3x - 17 = 0$$

It's obvious that there's no solution using the sum-and-product method: the only pairs of integers that multiply to give -17 are 1 and -17 and -1 and 17 , and neither pair adds up to 3 . If this happens, we can solve the equation by completing the square.

Example 1: Solve: $5x^2 - x - 10 = 0$

Solution: Using the ac method, we need two numbers that multiply to give -50 and add up to -1 . No two integers do that, so we can complete the square.

[1] Move the constant to the other side of the equal sign. We won't be using it, so let's get it out of the way.

$$\begin{aligned} 5x^2 - x - 10 &= 0 \\ 5x^2 - x &= 10 \end{aligned}$$

[2] If there's a leading coefficient on the x^2 term, factor it out. This makes the problem simpler by reducing it to an x^2 ... question, rather than an ax^2 ... question. This may create fractions, but if the original polynomial doesn't factor neatly, there are going to be fractions in the answer:

$$\begin{aligned} 5x^2 - x &= 10 \\ 5(x^2 - \frac{1}{5}x) &= 10 \end{aligned}$$

[3] Determine what third term could go into the brackets to make a perfect trinomial square. Take the coefficient in front of the x -term, divide it by two, and then square the result. If you're working with a fraction, the easiest way to divide by two is to double the denominator.

$$\begin{aligned} \frac{1}{5} \div 2 &= \frac{1}{10} \text{ [*] remember this number!} \\ (\frac{1}{10})^2 &= \frac{1}{100} \end{aligned}$$

Pay attention to that asterisked number, $\frac{1}{10}$. We'll need it later....



[4] Add and subtract this new term inside the bracket to make a perfect square.

You may have been told to add the new term to both sides, but in a problem like the one we're working on, that won't work because of the leading term that was factored out. Doing it all inside the bracket will help us to remember to include step [5]:

$$\begin{aligned}5(x^2 - \frac{1}{5}x) &= 10 \\5(x^2 - \frac{1}{5}x + \frac{1}{100} - \frac{1}{100}) &= 10\end{aligned}$$

[5] Move the subtracted term out of the bracket and over to the other side. The $-\frac{1}{100}$ is in the bracket with the 5 coefficient in front of it. To move it out of the bracket, we need to multiply it by the 5. This will almost always mean we can cancel:

$$\begin{aligned}5(x^2 - \frac{1}{5}x + \frac{1}{100} - \frac{1}{100}) &= 10 \\5(x^2 - \frac{1}{5}x + \frac{1}{100}) - \frac{5}{100} &= 10 \\5(x^2 - \frac{1}{5}x + \frac{1}{100}) - \frac{1}{20} &= 10 \\5(x^2 - \frac{1}{5}x + \frac{1}{100}) &= 10 + \frac{1}{20} \\5(x^2 - \frac{1}{5}x + \frac{1}{100}) &= 20\frac{1}{20}\end{aligned}$$

[6] "Factor" the perfect square and solve. Now we want to express the bracket as a binomial squared, but we don't have to do much work to factor it. We asked you to remember a number, $\frac{1}{10}$. That will be the number that completes the square. The sign in front of the x-term will tell you whether it should be $(x + \frac{1}{10})^2$ or $(x - \frac{1}{10})^2$:

$$\begin{aligned}5(x^2 - \frac{1}{5}x + \frac{1}{100}) &= 20\frac{1}{20} \\5(x - \frac{1}{10})^2 &= 20\frac{1}{20}\end{aligned}$$

We have now solved the problem of having both x^2 -terms and x-terms, so we can isolate x to solve. Taking a square root requires accounting for positive and negative possible solutions:

$$\begin{aligned}5(x - \frac{1}{10})^2 &= 20\frac{1}{20} \\(x - \frac{1}{10})^2 &= 20\frac{1}{100} \\x - \frac{1}{10} &= \pm \frac{\sqrt{201}}{10} \\x &= \frac{1 \pm \sqrt{201}}{10}\end{aligned}$$

WHEN AND WHY

The structure of that answer should look familiar if you've covered the quadratic formula, and for good reason. If you take the generic quadratic, $ax^2 + bx + c = 0$, and complete the square, you get the quadratic formula. For this reason, don't bother completing the square unless you are specifically asked to. (In this case check your work with the quadratic formula — completing the square is long and error-prone.) Later, when you cover conic sections and graphing circles, ellipses and sideways parabolas, completing the square will get the equation into the correct form. Keep this worksheet handy for when you start that section in Math 12!

Otherwise, if you're just solving a quadratic and the sum-and-product method doesn't give you a solution, go straight to the quadratic formula.



EXERCISES

A. Solve the following problems by completing the square.

1) $x^2 + 8x + 15 = 0$

3) $x^2 - 2x - 48 = 0$

2) $2x^2 - 9x + 7 = 0$

4) $2x^2 + 3x - 20 = 0$

B. Solve the problems in part A by using either the sum-and-product method or by the ac method, as appropriate.

C. Solve the following problems by completing the square.

1) $x^2 + 7x - 10 = 0$

3) $3x^2 + 13x + 9 = 0$

2) $2x^2 - 5x - 11 = 0$

4) $10x^2 - 16x - 17 = 0$

D. Solve the problems in part C by using the quadratic formula.

SOLUTIONS

A: (1) $x = -5$ or -3 (2) $x = 1$ or $7/2$ (3) $x = -6$ or 8 (4) $x = -4$ or $5/2$

B: As above.

C: (1) $x = \frac{-7 \pm \sqrt{89}}{2}$ (2) $x = \frac{5 \pm \sqrt{113}}{4}$ (3) $x = \frac{-13 \pm \sqrt{61}}{6}$ (4) $x = \frac{8 \pm 3\sqrt{26}}{10}$

D: As above.

