Rationalizing the Denominator

We always simplify our answers in math. This is so that the answer is in a format that is as easy to understand as possible. After all, $\frac{744}{1736}$ is hard to understand, but when we reduce it to $\frac{3}{7}$, it's easier to understand.

When radicals start showing up in fractions, things get worse. It's much harder to be able to tell whether two fractions containing radicals are equal. Consider the two fractions on the right. There's no obvious way to reduce them in any traditional sense, not like the example from the first paragraph, but they are in fact equal to each other. If you don't believe this, you can put both fractions into your calculator (with brackets around the numerators and denominators!) and you'll see they're both 2.414213562…

$$\frac{1 + 2\sqrt{2}}{3 - \sqrt{2}} \qquad \frac{6 + 5\sqrt{2}}{4 + \sqrt{2}}$$

So how do we simplify fractions like these? We can clear up ambiguous cases like this if we forbid putting radicals (or, spoiler alert, anything other than integers) from being in the denominator. This requires that we have some way of removing radicals from the bottom of the fraction, or rationalizing the denominator. There are a couple of things we may face.

Example 1: Simplify: $\frac{4 - 3\sqrt{5}}{\sqrt{7}}$

Solution: This fraction has just a square root in the denominator. We can raise this fraction to higher terms by multiplying through by $\sqrt{7}$:

$$\frac{4 - 3\sqrt{5}}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{4\sqrt{7} - 3\sqrt{35}}{7}$$

It makes the numerator a little bit worse, but that's the price we pay for clarity. This is the simplified form. With a square root, we multiply again by the square root we see to fix the problem. (And only the square root! if the denominator is something like $2\sqrt{3}$, we need only multiply by $\sqrt{3}$ to solve the problem.) Things are a little different if the denominator is not a square root.

Example 2: Simplify: $\frac{6xy^2 + 3y\sqrt[3]{x}}{\sqrt[3]{18x^2y}}$

Solution: We can't use quite the same technique for this problem. Multiplying through by the denominator again will not make the radicand a perfect cube so we can get rid of the radical sign in the denominator. We could multiply by $\sqrt[3]{(18x^2y)^2}$, but that's not the lowest answer that does the job. It will make a mess that we'll only need to clean up later. The best practice is to figure out what new radicand will make all the factors in

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the existing radicand perfect cubes, i.e., to make their exponents a multiple of 3. This includes breaking the coefficient 18 down into its prime factors.

The number 18 breaks down to 2×3^2. The exponent on 2 needs to be increased by 2 to become 3; the exponent on 3 needs to be increase by 1 to become 3. Likewise the exponent on x needs one more and the exponent on y needs two more. Our best choice of a cube root radicand to multiply in is $2^2 \times 3 \times xy^2 = 12xy^2$:

$$\frac{6xy^2 + 3y\sqrt[3]{x}}{\sqrt[3]{18x^2y}} \times \frac{\sqrt[3]{12xy^2}}{\sqrt[3]{12xy^2}} = \frac{6xy^2\sqrt[3]{12xy^2} + 3y\sqrt[3]{12x^2y^2}}{\sqrt[3]{216x^3y^3}}$$

$$= \frac{6xy^2\sqrt[3]{12xy^2} + 3y\sqrt[3]{12x^2y^2}}{6xy}$$

$$= \frac{2xy\sqrt[3]{12xy^2} + \sqrt[3]{12x^2y^2}}{2x}$$

The final possibility is if the denominator is a binomial, either with an integer and a radical, or two distinct radicals. We'd like to find some way to square each one without creating the two terms in the middle that you get from FOILing. A difference of squares does this nicely.

$$(a + b)(a - b) = a^2 - b^2$$

We just need to multiply by the other factor that creates a difference of squares. There's a name for that other factor — it's called the **conjugate**. It's the expression that is the same as the denominator we wish to fix, but with its plus sign replaced with a minus sign, or vice versa.

Example 3: Simplify: $\frac{1 + 2\sqrt{2}}{3 - \sqrt{2}}$. (This is one of the two equal fractions from page 1.)

Solution: The denominator has a minus sign, so the conjugate has a plus sign:

$$\frac{1 + 2\sqrt{2}}{3 - \sqrt{2}} \times \frac{3 + \sqrt{2}}{3 + \sqrt{2}} = \frac{3 + 6\sqrt{2} + \sqrt{2} + 4}{9 + 3\sqrt{2} - 3\sqrt{2} - 2} = \frac{7 + 7\sqrt{2}}{7} = 1 + \sqrt{2}$$

On the other hand, just because there's a plus sign or minus sign in the denominator, that doesn't mean we want the conjugate:

Example 4: Simplify: $\frac{8}{\sqrt{x - 4}}$.

Solution: Be careful! The denominator here is *not* a binomial. It's one term, so we should use the technique from Example 2:

$$\frac{8}{\sqrt{x - 4}} \times \frac{\sqrt{x - 4}}{\sqrt{x - 4}} = \frac{8\sqrt{x - 4}}{x - 4}$$

EXERCISES

A. Simplify.

1) $\dfrac{7\sqrt{3}}{\sqrt{2}}$ 4) $\dfrac{2-4\sqrt{3}}{3\sqrt{2}}$ 7) $\dfrac{7-2\sqrt{3}+9\sqrt{7}}{\sqrt{5}}$

2) $\dfrac{12\sqrt{7}}{\sqrt{3}}$ 5) $\dfrac{8+\sqrt{12}}{\sqrt{5}}$ 8) $\dfrac{21+7\sqrt{6}+\sqrt{15}}{2\sqrt{3}}$

3) $\dfrac{1+\sqrt{5}}{\sqrt{6}}$ 6) $\dfrac{5\sqrt{6}-3\sqrt{10}}{\sqrt{8}}$ 9) $\dfrac{\sqrt{98}-\sqrt{96}}{\sqrt{3}}$

B. Simplify. You may assume that none of these variables are taking on negative values, and so your answers do not require absolute value signs.

1) $\dfrac{\sqrt[3]{7}}{\sqrt[3]{2}}$ 4) $\dfrac{\sqrt[4]{24ab^3}}{\sqrt[4]{32a^3b^2}}$ 7) $\dfrac{\sqrt{x}}{\sqrt[3]{9}}$

2) $\dfrac{\sqrt[3]{x}}{\sqrt[3]{9}}$ 5) $\dfrac{3}{\sqrt[4]{126t^3}}$ 8) $\dfrac{\sqrt[15]{216s^3}}{\sqrt[5]{18s^4}}$

3) $\dfrac{\sqrt[3]{m^2}+\sqrt[3]{n}}{\sqrt[3]{mn^2}}$ 6) $\dfrac{\sqrt[4]{375p^2}-\sqrt[4]{1500pq^2}}{2\sqrt[4]{5400p^3q}}$ 9) $\dfrac{\sqrt{3k}-\sqrt[3]{3h}}{\sqrt[6]{9h^4k}}$

C. Simplify. You may assume that none of these variables are taking on negative values, and so your answers do not require absolute value signs.

1) $\dfrac{6+5\sqrt{2}}{4+\sqrt{2}}$ 4) $\dfrac{y+7}{\sqrt{y+3}}$ 7) $\dfrac{6}{1+\sqrt[3]{2}}$

[*Hint:* With square roots, we made a difference of squares…]

2) $\dfrac{-11-\sqrt{2}}{3-10\sqrt{2}}$ 5) $\dfrac{\sqrt{6}-\sqrt{5}}{3\sqrt{5}-\sqrt{3}}$

3) $\dfrac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$ 6) $\dfrac{5+\sqrt{3}-2\sqrt{2}}{3-\sqrt{6}}$ 8) $\dfrac{2\sqrt{3}}{x-1+\sqrt{3}}$

[*Hint:* Group (x − 1).]

SOLUTIONS

A: (1) $\frac{7\sqrt{6}}{2}$ (2) $4\sqrt{21}$ (3) $\frac{\sqrt{6}+\sqrt{30}}{6}$ (4) $\frac{\sqrt{2}-2\sqrt{6}}{3}$ (5) $\frac{8\sqrt{5}-2\sqrt{15}}{5}$ (6) $\frac{5\sqrt{3}-3\sqrt{5}}{2}$
(7) $\frac{7\sqrt{5}-2\sqrt{15}+9\sqrt{35}}{5}$ (8) $\frac{7\sqrt{3}-7\sqrt{2}+\sqrt{5}}{2}$ (9) $\frac{7\sqrt{6}-12\sqrt{2}}{3}$

B: (1) $\frac{\sqrt[3]{28}}{2}$ (2) $\frac{\sqrt[3]{3x}}{3}$ (3) $\frac{m\sqrt[3]{mn}+\sqrt[3]{m^2n^2}}{mn}$ (4) Simplify first: $=\sqrt[4]{\frac{3b}{4a^2}}=\frac{\sqrt[4]{12a^2b}}{2a}$ (5) $\frac{\sqrt[4]{2^3 3^2 7^3 t}}{14t}$
(6) $\frac{\sqrt[4]{90p^3q^3}+\sqrt[4]{360p^3q}}{12pq}$ (7) $\frac{\sqrt[6]{9x^2}}{3}$ (8) Reduce index in numerator: $=\frac{(6^3s^3)^{1/15}}{\sqrt[5]{18s^4}}=\sqrt[5]{\frac{6s}{18s^4}}=\frac{1}{\sqrt[5]{3s^3}}=\frac{\sqrt[5]{3^4s^2}}{3s}$
(9) Raise indices in numerator: $=\frac{\sqrt[6]{27k^3}+\sqrt[6]{9h^2}}{\sqrt[6]{9h^4k}}=\frac{k\sqrt[6]{3h^2k^2}+\sqrt[6]{9h^4k^5}}{hk}$

C: (1) $1+\sqrt{2}$ (2) $1+\sqrt{2}$ (3) $\frac{x+2\sqrt{7x}+7}{x-7}$ (4) $\frac{(y+7)\sqrt{y+3}}{y+3}$ (5) $\frac{3\sqrt{2}-\sqrt{15}+3\sqrt{30}+15}{12}$
(6) $\frac{15+5\sqrt{6}-\sqrt{3}-3\sqrt{2}}{3}$ (7) $2-\sqrt[3]{2}+\sqrt[3]{4}$ (8) Conjugate is: $=(x-1)-\sqrt{3}$; $\frac{2x\sqrt{3}-2\sqrt{3}-6}{x^2-2x-2}$