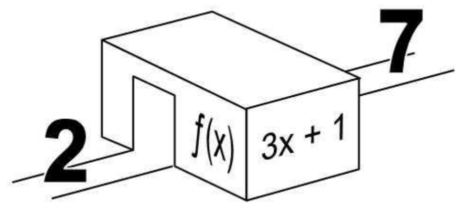
VANCOUVER COMMUNITY COLLEGE

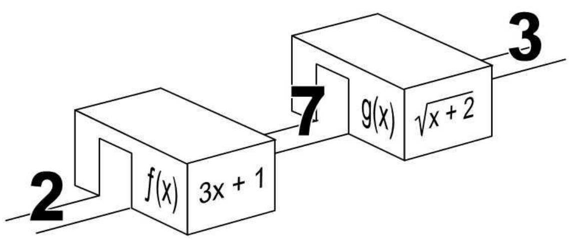
Composite Functions

A function is nothing more than a set of instructions. We usually write these instructions as a formula, so the instructions, "Take a number, multiply it by 3 and then add 1" would be written $f(x) = 3x + 1$.

We can visualize the function as a workshop: We can visualize the function as a workshop: numbers come in, they get worked on, and new numbers come out. (And, of course, if we're discussing a function, for any number that can be input into the function, there cannot be any ambiguity as to which number comes out.)



A **composite function** is a function where the input from one function comes from the output of another function. There are a couple of ways that we can write this. If we have another function $g(x) = \sqrt{x + 2}$, then we can write the set-up illustrated at the left as $g(f(x))$, meaning that the output of the function $f(x)$ becomes the input of the function $g(x)$. So:



$$g(f(2)) = g(3 \cdot 2 + 1) = g(7)$$
$$= \sqrt{7 + 2} = \sqrt{9} = 3$$

We can also write this composite function as $(g \circ f)(x)$, which we read as "g of f of x".

Notice that the order of the functions matters! If we put the functions in the other order, we get:

$$(f \circ g)(2) = f(\sqrt{2 + 2}) = f(\sqrt{4}) = f(2)$$
$$= 3 \cdot 2 + 1 = 7$$

Besides plugging numbers into the formulas, we can also determine what single function does the job of the two composed functions together. We can do this by composing the function definitions:

$$(g \circ f)(x) = g(f(x)) = g(3x + 1)$$
$$= \sqrt{(3x + 1) + 2} = \sqrt{3x + 3}$$

Check: $(g \circ f)(2) = \sqrt{3 \cdot 2 + 3} = \sqrt{9} = 3$ ✓

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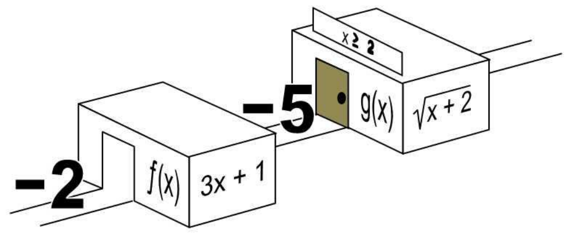
DOMAIN OF A COMPOSITE FUNCTION

We're also interested in what numbers are in the domains of composite functions. Recall that the domain of a function is the list of all numbers that give a real number for an answer.

It should be obvious that any number that is outside the domain of $f(x)$ alone should be outside the domain of the composite function $g(f(x))$. If we can't get a real number output from $f(x)$, we don't get anything to plug into g.

What if, on the other hand, the "second" function, g has a restriction on its domain? Then it's easily possible that we could plug a number into f which results in an output that is outside the domain of g. For the example we've been looking at, the domain of g is $x \geq -2$, so $(g \circ f)(-2)$ is undefined because we can't complete the second calculation.



For $(g \circ f)(x)$ to be defined, $f(x)$ must be within the domain of g:

$$f(x) \geq -2$$
$$3x + 1 \geq -2$$
$$3x \geq -3$$
$$x \geq -1$$

We've already determined that $(g \circ f)(x) = \sqrt{3x + 3}$, and if we analyse it, we'll see that the domain of this version of the function is also D: $\{x \geq -1\}$. So can we just compose the functions and find the domain of the overall function this way? Most of the time, but we have to be careful! Consider this simple example:

$$h(x) = x^2$$
$$k(x) = \sqrt{x}$$

Then: $$(h \circ k)(x) = h(k(x)) = (\sqrt{x})^2 = x$$

…and that function doesn't have any restrictions on its domain. So is the domain of $h \circ k$ all real numbers? No! Our "inside" function, $k(x)$, still has a restriction. The composite function is defined as having two steps: take the square root of x, and then square the result. If $x = -1$, we can't perform the first step, so the composite function is not defined. The domain restriction on $k(x)$ carries through to the composite function:

$$(h \circ k)(x) = x, \text{ where } x \geq 0$$

In summary, the following numbers are outside the domain of a composite function $g \circ f$:

- Any number that is outside the domain of the "inside" function, f, and
- Any number x that generates an output $f(x) = n$, where n is outside the domain of g

REWRITING A FUNCTION AS A COMPOSITE FUNCTION

Frequently your textbook will give you questions like the following when you're covering composite functions.

Example 1a: Express the following function as a composite function: $F(x) = \sin(x^2 - 5)$.

There is no single correct answer to this question. (There's a *best* answer, but we'll get to that in a minute.) If $f(x) = \sin(x^2 - 5) + 1000$ and $g(x) = x - 1000$, then $g(f(x)) = F(x)$ are a valid answer to this question, but it's not the answer your instructor or the textbook is hoping for. What we really mean by this question is:

Example 1b: Express the following function as a composite function *of two elementary functions*: $F(x) = \sin(x^2 - 5)$.

Elementary functions are functions that only feature one type of calculation. Broadly, they are "algebraic" functions (polynomials), rational functions (big fractions), root functions, trigonometric functions, inverse trigonometric functions, exponential functions and logarithmic functions. If you don't recognize some of these, don't worry, you'll know them all by the time you're done Math 12. We're getting you ready for calculus, where you'll need to pick a function apart into simple pieces to work with it.

Solution: This function is the sine of a polynomial, so the solution is $f(x) = x^2 - 5$ and $g(x) = \sin x$. Then $g(f(x)) = F(x)$.

Example 2: Given $f(x) = x - 2$, and $F(x) = 7x + 5$, find a function g so that $(g \circ f)(x) = F(x)$ for all x.

Solution: The easiest way to solve this kind of problem is to start with $f(x)$ and apply operations to it until it looks like $F(x)$. We're starting from $x - 2$, and we'd like something that starts with 7x… so we probably want to multiply $f(x)$ by 7:

$$
\begin{aligned}
f(x) &= x - 2 \\
7(f(x)) &= 7(x - 2) \\
&= 7x - 14 \\
7(f(x)) + 19 &= 7x - 14 + 19 \\
&= 7x + 5 = F(x) \\
\therefore \quad g(f(x)) &= 7(f(x)) + 19 \\
g(x) &= 7x + 19
\end{aligned}
$$

EXERCISES

Some clearly marked questions are for MATH 1020 students only.

A. Find $(f \circ g)(2)$ and $(g \circ f)(-5)$ for each of the following pairs of functions.

1) $f(x) = x^2 + 4$, $g(x) = 2x - 3$
2) $f(x) = x + 6$, $g(x) = 2^x$
3) $f(x) = \frac{1}{x + 8}$, $g(x) = -4x$
4) $f(x) = \sqrt{x + 9}$, $g(x) = -3x + 1$
5) $f(x) = -3x + 5$, $g(x) = \frac{1}{\sqrt{3x + 3}}$
6) $f(x) = \frac{1}{\sqrt{3x + 3}}$, $g(x) = -3x + 5$

B. For each pair of functions, find a function $F(x)$ so that $F(x) = (g \circ f)(x)$. State the domain of $F(x)$.

1) $f(x) = x + 5$, $g(x) = x^2$
2) $f(x) = 3x + 2$, $g(x) = \frac{1}{2}x - 7$
3) $f(x) = x + 1$, $g(x) = 2^x$
4) $f(x) = 2x - 1$, $g(x) = 10x^2 - x + 7$
5) $f(x) = x^2 + 1$, $g(x) = \sqrt{x} - 1$

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6) $f(x) = \cos x$, $g(x) = 1/x$
7) $f(x) = \tan x$, $g(x) = x^3 + x$

8) $f(x) = \sqrt{x^2 - 9}$, $g(x) = x^2 + 8$
9) $f(x) = \frac{x+2}{x-4}$, $g(x) = \frac{1}{x+1}$
10) $f(x) = 17^{(x+3)} + x^{57}$, $g(x) = 12$

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C. For each function, find elementary functions $f(x)$ and $g(x)$ so that $F(x) = (g \circ f)(x)$.

1) $F(x) = \sin(x^3 - 5x)$
2) $F(x) = 6^{(x^2 - 7)}$
3) $F(x) = \sqrt{\sec x}$
4) $F(x) = (\log x)^2 + 6 \log x + 9$
5) $F(x) = \cos^2 x$
6) $F(x) = 4 \tan x + \cot x$

D. For each pair of functions, find a function $g(x)$ so that $F(x) = (g \circ f)(x)$.

1) $F(x) = x - 7$, $f(x) = x + 7$
2) $F(x) = 8x + 10$, $f(x) = x + 2$
3) $F(x) = \frac{x+5}{3}$, $f(x) = 2x + 9$
4) $F(x) = \sin(x + 1) - \cos(x + 1)$, $f(x) = x + 1$
5) $F(x) = \sqrt{x^2 - 9}$, $f(x) = x^2 + 4$
6) $F(x) = x^2 + 4x - 8$, $f(x) = x + 2$

SOLUTIONS

A: (1) $(f \circ g)(2) = 5$; $(g \circ f)(-5) = 55$ (2) 10; 2 (3) undefined; $-\frac{4}{3}$ (4) 2, −5
(5) 4; $\frac{\sqrt{63}}{63}$ (6) undefined; undefined

B: (1) $F(x) = x^2 + 10x + 25$, D: $\mathbb{R}$ (2) $F(x) = \frac{3}{2}x^2 - 6$, D: $\mathbb{R}$ (3) $F(x) = 2^{(x+1)}$, D: $\mathbb{R}$
(4) $F(x) = 40x^2 - 42x + 18$, D: $\mathbb{R}$ (5) $F(x) = \sqrt{x^2 + 1} - 1$, D: $\mathbb{R}$
(6) $F(x) = \sec x$, D: $\{x \in \mathbb{R} \mid x \neq (2k + 1)(\frac{\pi}{2})$, k is an integer$\}$
(7) $F(x) = \tan^3 x + \tan x$, D: $\{x \in \mathbb{R} \mid x \neq (2k + 1)(\frac{\pi}{2})$, k is an integer$\}$
(8) $F(x) = x^2 - 1$, D: $\{x \in \mathbb{R} \mid x \leq -3 \text{ or } x \geq 3\}$
(9) $F(x) = \frac{x-4}{2x-2}$, D: $\{x \in \mathbb{R} \mid x \neq 1 \text{ and } x \neq 4\}$ (10) $F(x) = 12$, D: $\mathbb{R}$

C: (1) $f(x) = x^3 - 3$, $g(x) = \sin x$ (2) $f(x) = x^2 - 7$, $g(x) = 6^x$ (3) $f(x) = \sec x$, $g(x) = \sqrt{x}$
(4) $f(x) = \log x$, $g(x) = x^2 + 6x + 9$ (5) $f(x) = \cos x$, $g(x) = x^2$
(6) $f(x) = \tan x$, $g(x) = 4x + \frac{1}{x}$; or $f(x) = \cot x$, $g(x) = \frac{4}{x} + x$

D: (1) $g(x) = x - 14$ (2) $g(x) = 8x - 6$ (3) $g(x) = \frac{x+1}{6} = \frac{1}{6}x + \frac{1}{6}$
(4) $g(x) = \sin x - \cos x$ (5) $g(x) = \sqrt{x - 13}$ (6) $g(x) = x^2 - 12$