VANCOUVER COMMUNITY COLLEGE

Higher-Order Polynomials

FACTOR THEOREM If, for a polynomial $f(x)$, the value of $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

REMAINDER THEOREM For a polynomial $f(x)$, the value of $f(a)$ is the remainder when $f(x)$ is divided by $(x - a)$.

RATIONAL ZEROES TH'M The rational zeroes of a polynomial with integer coefficients will have a factor of the constant term as the numerator and a factor of the leading coefficient as the denominator.

EXERCISES

A. Determine the remainder after each of the following divisions:

1) $(x^4 - 3x^3 + 5x + 8) \div (x + 1)$
3) $(x^{17} + 1) \div (x - 1)$
2) $(x^8 - x^5 - x^3 + 1) \div (x + i)$
4) $(2x^5 - 7) \div (x + 1)$

B. Show that $(x - 3)$ is a factor of the polynomial $f(x) = x^4 - 4x^3 - 7x^2 + 22x + 24$.

C. Evaluate:

1) $P(-2)$ given that $P(x) = x^3 + x + 10$
2) $f(2i)$ given that $f(x) = x^3 + 3x^2 - 3x + 6$

D. Determine if the following are roots of the given equation:

1) -1, $x^3 - 7x - 6 = 0$
3) $2i$, $2x^3 + 3x^2 + 8x + 12 = 0$
2) 2, $x^4 - 2x^2 - x + 7 = 0$
4) 2, $x^5 - 6x^3 + 4x^2 - x + 4 = 0$

E. Determine the value(s) of k such that:

1) $(x - 2)$ is a factor of $2x^3 + 3x^2 - kx + 10$
2) $4x^3 + 3x^2 - kx + 6k$ is exactly divisible by $(x + 3)$
3) $2x^3 - kx^2 + 6x - 3k$ is exactly divisible by $(x + 2)$
4) $(x^4 - k^2x + 3 - k) \div (x - 3)$ has a remainder of 4
5) 2 is a zero of the polynomial $f(x) = x^5 + 4kx - 4k^2$

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6) the equation $f(x) = 2x(x + 1)(x^2 + k)$ has zeroes at 0, −1, and $\pm 2i$.

7) 2 is a root of the equation $x^3 - 9x + k = 0$

F. Given that $f(x) = x^3 - 6x^2 - 2x + 40$, use synthetic division to evaluate:

1) $f(-5)$

2) $f(4)$

G. Given the following roots, solve the equation for the other roots:

1) $x^3 + 2x^2 - 23x - 60 = 0$; 5 is a root

2) $x^4 - 2x^2 - 3x - 2 = 0$; −1 and 2 are roots

H. Determine the roots of the following equations:

1) $(x - 1)^2(x + 2)(x + 4) = 0$

2) $x^3(x^2 - 2x - 15) = 0$

3) $x^2 - ix + 12$

I. Determine the simplest equation having only the following roots:

1) 5, 1, −3

2) $\pm 2,\ 2 \pm \sqrt{3}$

3) $0,\ 1 \pm 5i$

J. Determine the simplest equation with *integer* coefficients having only the following roots:

1) $\pm 3i,\ \pm \frac{1}{2}\sqrt{2}$

b) −1, 2 (triple root)

K. The following are roots of equations with *real* coefficients. What other number must also be a root in each case?

1) $2i$

2) $-3 + 2i$

3) $-3 - i\sqrt{2}$

L. The following are roots of equations with *rational* coefficients. What other number must also be a root in each case?

1) $-\sqrt{7}$

2) $-4 + 2\sqrt{3}$

M. Are the following statements valid?

1) If $x = i$ is a root of $x^3 + 7x - 6i = 0$, then $-i$ must also be a root.

2) If $\sqrt{3} - i\sqrt{2}$ is a root of $x^3 + (1 - 2\sqrt{3})x^2 + (5 - 2\sqrt{3})x + 5 = 0$, then $\sqrt{3} + i\sqrt{2}$ must also be a root.

3) If $-1 + \sqrt{2}$ is a root of $x^4 + (1 - 2\sqrt{2})x^3 + (4 - 2\sqrt{2})x^2 + (3 - 4\sqrt{2})x + 1 = 0$, then $-1 - \sqrt{2}$ must also be a root.

N. Write the simplest equation of lowest degree with satisfies the following:

1) *real* coefficients; 2 and $1 - 3i$ are some of the roots

2) *rational* coefficients; $-1 + \sqrt{5}$ and -6 are two of the roots

3) *rational* coefficients; $-5i$ and $\sqrt{6}$ are two of its roots

4) *rational* coefficients; $2 + i$ and $1 - \sqrt{3}$ are two of its roots

5) *complex* coefficients; 2 and $1 - 3i$ are the only roots

6) *real* coefficients; -6 and $-1 + \sqrt{5}$ are the only roots

O. Determine the four roots of $x^4 + 2x^2 + 1 = 0$.

P. Determine all rational roots, if any:

1) $x^4 - 2x^2 - 3x - 2 = 0$

3) $2x^3 + x^2 - 7x - 6 = 0$

2) $x^3 - x - 6 = 0$

4) $2x^4 + x^2 + 2x - 4 = 0$

Q. Solve:

1) $x^3 - 2x^2 - x + 2 = 0$

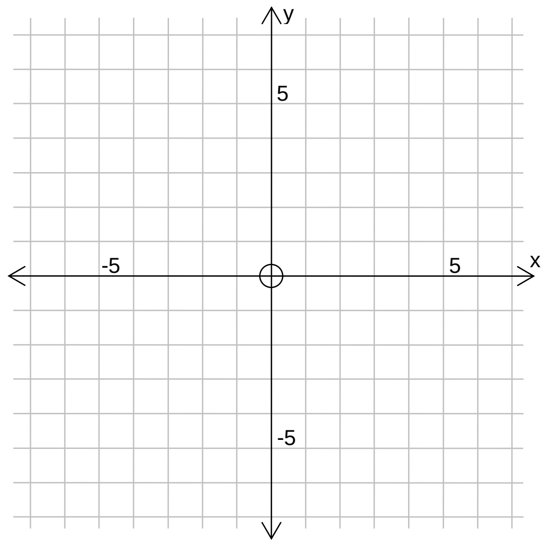
3) $2x^4 - 3x^3 - 7x^2 - 8x + 6 = 0$

2) $x^3 - 2x^2 - 31x + 20 = 0$

R. Graph $f(x) = \frac{1}{10}(x^5 - 3x^4 - 7x^3 + 25x^2 + 8x - 60)$. From the graph, determine:

1) the number of positive, negative and complex roots

2) the values of any real root



S. According to Descartes' Rules, how many complex roots does the equation $2x^5 + x^4 + x^2 + 6 = 0$ have?

T. How many *real* roots does the equation $x(x^2 + 1)(x^2 - 1) = 0$ have?

SOLUTIONS

A. (1) 7 (2) 2 (3) 2 (4) −9 B. $f(3) = 0$ C. (1) $P(-2) = 0$ (2) $f(2i) = -6 - 14i$
D. (1) Yes. (2) No. (3) Yes. (4) No. E. (1) k = 19 (2) k = 9 (3) k = −4
(4) $k = -\frac{16}{3}$, 5 (5) k = −2, 4 (6) k = 4 (7) k = 10 F. (1) −225 (2) 0
G. (1) −3 and −4 (2) $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ H. (1) −4, −2 and 1 (×2) (2) 0 (×3), −3, 5 (3) $-3i$, $4i$
I. (1) $(x - 5)(x - 1)(x + 3) = 0 \rightarrow x^3 - 3x^2 - 13x + 15 = 0$ (2) $(x^2 - 4)(x^2 - 4x + 1) = 0 \rightarrow x^4 - 4x^3 - 3x^2 + 16x - 4 = 0$ (3) $x(x^2 - 2x + 26) = 0 \rightarrow x^3 - 2x^2 + 26x = 0$
J. (1) $(x^2 + 9)(x^2 - \frac{1}{2})(2) = 0 \rightarrow 2x^4 + 17x^2 - 9 = 0$ (2) $(x - 2)^3(x + 1) = 0 \rightarrow x^4 - 5x^3 + 6x^2 + 4x - 8 = 0$ K. (1) $-2i$ (2) $-3 - 2i$ (3) $-3 + i\sqrt{2}$ L. (1) $\sqrt{7}$ (2) $-4 + 2\sqrt{7}$ M. (1) Not necessarily. (In fact, i is a root and $-i$ is not.) (2) Yes, since the coefficients are real. (3) Not necessarily. (In fact, $-1+\sqrt{2}$ is a root, and $-1-\sqrt{2}$ isn't.) N. (1) $(x - 2)(x^2 - 2x + 10) = 0 \rightarrow x^3 - 4x^2 + 14x - 20 = 0$ (2) $(x + 6)(x^2 + 2x - 4) = 0 \rightarrow x^3 + 8x^2 + 8x - 24 = 0$ (3) $(x^2 + 25)(x^2 - 6) = 0 \rightarrow x^4 + 19x^2 - 150 = 0$ (4) $(x^2 - 4x + 5)(x^2 - 2x - 2) = 0 \rightarrow x^4 - 6x^3 + 11x^2 - 2x - 10 = 0$ (5) $x^2 + (-3 + 3i)x + 2 - 6i = 0$ (6) $x^2 + (7 - \sqrt{5})x + 6 - 6\sqrt{5} = 0$ O. $\pm i$, both double
P. (1) −1, 2 (2) 2 (3) $-\frac{3}{2}$, −1, 2 (4) No real roots.
Q. (1) x = −1, 1, 2 (2) −5, $\frac{7}{2} \pm \frac{\sqrt{33}}{2}$ (3) ½, 3, $-1 \pm i$
R. (1) 1 positive, 2 negative, 2 complex (2) −2 (double), 3
S. There is one negative real root, and no positive real roots, so there must be 4 complex roots.
T. 3 (namely, 0, 1 and −1)