Logarithms

A **logarithm** is a way to express the relationship between numbers in an exponential expression.

   Every operation in mathematics has an opposite. The opposite of adding is subtracting. The opposite of multiplying is dividing. Taking a root (square root, cube root,…) is one opposite of an exponent. Look at this exponential expression along with its terminology. I can write an expression with 2 and 5 that gives the power, 32, as the answer:

$$\text{ANSWER} \Rightarrow 32 = 2^{5 \,\Leftarrow\, \text{EXPONENT}}_{\Leftarrow\, \text{BASE}}$$

We can rewrite this relationship as a radical expression:

$$\text{INDEX} \Rightarrow {}^{5}\!\!\sqrt{\underset{\text{RADICAND}}{32}} = 2 \Leftarrow \text{ROOT}$$

The positions of the numbers have been changed, but it's the same relationship. The radical expression has the 5 and the 32. The base from the original equation, 2, is now the "answer" in the equation.

   Neither of these expressions can use the 32 and 2 to give an answer of 5, the exponent from the original equation. That's what logarithms do. When you see a logarithm expression, it asks the question, **"What's the missing exponent?"** Logarithms are another way of writing exponents.

$$\underset{\text{BASE}}{\overset{\text{ANSWER}}{\log_2 32}} = 5 \Leftarrow \text{LOGARITHM}$$

Example 1:   Express as a logarithm: $6^3 = 216$ and $\sqrt[7]{x} = y$.

Solution: Logarithms ask the question, "what is the exponent?" This will help us remember where each number goes:

$6^3 = 216$ is equivalent to $\log_6 216 = 3$, and
$\sqrt[7]{x} = y$ is equivalent to $\log_x y = \frac{1}{7}$.

We could also interpret the expressions as $\sqrt[3]{216} = 6$, and so $\log_{216} 6 = \frac{1}{3}$ and $y^7 = x$, or $\log_y x = 7$. It's better to just use what you see.

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Just as there are laws for exponents and radicals, there are also laws for logarithms.

$\log_a (a^x) = x$ $\log_x x^4 = 4$; $\log_7 49 = \log_7 7^2 = 2$; $\log_m m = 1$

$a^{\log_a x} = x$ $5^{\log_5 12} = 12$

$\log_a (xy) = \log_a x + \log_a y$ $\log_{10} 100x = \log_{10} 100 + \log_{10} x = 2 + \log_{10} x$

$\log_a (x/y) = \log_a x - \log_a y$ $\log_2 \frac{3}{8} = \log_2 3 - \log_2 8 = \log_2 3 - 3$

$\log_a (x^y) = y \log_a x$ $\log_5 4^6 = 6 \log_5 4$; $\ln e^x = x \ln e = x \cdot 1 = x$

$\log_a 1 = 0$ for any a $\log_{125} 1 = 0$

$\log_a -b$ is undefined for a, b > 0 $\log_3 -27$ is not a real number.

$\log_b M = \dfrac{\log_a M}{\log_a b}$ $\log_5 30 = \dfrac{\log_{10} 30}{\log_{10} 5} = 1.4771\ldots \div 0.6990 = 2.1133$

This last law, which is called the **change of base law**, is very useful since most calculators can only do logarithms with a base of 10 (**common logarithms**) or with a base *e* (**natural logarithms**). Mathematicians don't write a base for common logarithms, so "$\log_{10} 1000$" is usually written as just "log 1000". The symbol for a natural logarithm is "ln". (The abbreviation is short for *logarithmus naturalis*, Latin for "natural logarithm".) Instead of "$\log_e 5$", we write "ln 5".

Example 2: Evaluate: $\log_7 2401$ and $\log_2 35$.

Solution: Logarithms ask the question, "what is the exponent?" First, we need to know what exponent you put on 7 to get 2401. Since few people know this off the top of their heads, we can either guess and test, or use the change of base law.

$7^0 = 1$; $7^1 = 7$; $7^2 = 49$; $7^3 = 343$; $7^4 = 2401$; …

The answer to the first part is 4. We can tell by looking that the second one will not work. 35 is not a power of 2. (But $32 = 2^5$ is, so we expect an answer close to 5.) We must use the change of base law to evaluate the second expression:

$$\log_2 35 = \frac{\log 35}{\log 2} = \frac{1.54406\ldots}{0.30103\ldots} = 5.12928\ldots$$

Example 3: Express in terms of log x and log y: $\log xy^2$ and $\log \frac{x^2}{y}$.

Solution: We can use the laws of logarithms (especially the first three on this page) to simplify these expressions.

$$\log xy^2 = \log x + \log y^2 = \log x + 2 \log y$$
$$\log \frac{x^2}{y} = \log x^2 - \log y = 2 \log x - \log y$$

EXERCISES

A. Express in logarithmic form. Use the most obvious expression:

1) $19^3 = 6859$

3) $\sqrt{1369} = 37$

2) $a^b = c$

4) $\sqrt[5]{x^3} = y$

B. Express as exponential expressions or radicals, as appropriate:

1) $\log_5 625 = 4$

4) $\log_{128} 2 = \frac{1}{7}$

2) $\log_x 1 = 0$

5) $\log_q r = \frac{5}{4}$

3) $\log 50 = 1.7$

6) $\ln 3 = 1.1$

C. Evaluate without a calculator:

1) $\log_2 8$

7) $\log_3 \frac{1}{3}$

2) $\log 1{,}000{,}000$

8) $\log_4 \frac{1}{64}$

3) $\log \sqrt[3]{10}$

9) $\log_{12} (-144)$

4) $\ln e^{3.7}$

10) $\log_{0.5} 8$

5) $\log_7 (49^2)$

11) $\log_5 125\sqrt{5}$

6) $\log_5 625$

12) $\log_2 \sqrt[4]{1024}$

D. Evaluate to four decimal places using a calculator:

1) $\log (-12)$

4) $\log_4 123$

2) $\log 15$

5) $\log_7 343$

3) $\ln 2.33$

6) $\log_{100} 355$

E. Express in terms of log x, log y and log z:

1) $\log x^2 y^3 z^4$

3) $\log \dfrac{xy^2}{yz}$

2) $\log \dfrac{x^2}{y^3}$

4) $\log \sqrt{\dfrac{x}{yz}}$

F. Express as a single logarithm and simplify:
1) log x + ½ log y

3) log x − (½ log x + log y) − log y

2) 2 log x + 3(log y − log x)

4) 2 log x − 3 log y

G. Express as a single logarithm:
1) $\frac{1}{2}$ log 25 − $\frac{1}{3}$ log 64 + $\frac{2}{3}$ log 27

3) 2 log 3 + 4 log 2 − 3

2) log 5 − 1 [*not log (5 − 1)!*]

4) $\log_4$ 5 − $\log_3$ 9

H. Given that log 2 = 0.3010, log 3 = 0.4771 and log 7 = 0.8451, evaluate without your calculator's "log" button:
1) log 108

5) log 0.42

2) log 5

6) log 0.0081

3) log $\sqrt[3]{72}$

7) log 1500

4) log $\frac{1}{2}$

8) log 8.$\overline{3}$

SOLUTIONS

A. (1) $\log_{19}$ 6859 = 3 (2) $\log_a$ c = b (3) $\log_{1369}$ 37 = $\frac{1}{2}$ (4) $\log_x$ y = $\frac{3}{5}$

B. (1) 5^4 = 625 (2) x^0 = 1 (3) $10^{1.7}$ = 50 (4) $\sqrt[7]{128}$ = 2 (5) $\sqrt[t]{q^s}$ = r (6) $e^{1.1}$ = 3

C. (1) 3 (2) 6 (3) $\frac{1}{3}$ (4) 3.7 (5) 4 (6) 4 (7) −1 (8) −3 (9) undefined (10) −3
   (11) $\frac{7}{2}$ (12) $\frac{5}{2}$

D. (1) undefined (2) 1.1761 (3) 0.8459 (4) 3.4713 (5) 3 (6) 1.2751

E. (1) 2 log x + 3 log y + 4 log z (2) 2 log z − 3 log y (3) log x + log y − log z
   (4) ½ (log x − log y − log z)

F. (1) log x$\sqrt{y}$ (2) log $\frac{y^3}{x}$ (3) log $\frac{\sqrt{x}}{y^2}$ (4) log $\frac{x^2}{y^3}$

G. (1) log $\frac{45}{4}$ (2) log $\frac{1}{2}$ (3) log $\frac{18}{125}$ (4) $\log_4$ $\frac{5}{16}$

H. (1) log (3^3 × 2^2) = 3 log 3 + 2 log 2 = 2.0333 (2) log (10 ÷ 2) = log 10 − log 2
   = 1 − 0.3010 = 0.6990 (3) log ($72^{1/3}$) = ⅓ log (3^2 × 2^3) = ⅔ log 3 + log 2 = 0.6190$\overline{6}$
   (4) log 1 − log 2 = 0 − 0.3010 = −0.3010 (5) log $\frac{42}{100}$ = log (2 × 3 × 7) − log 100
   = (0.3010 + 0.4771 + 0.8451) − 2 = −0.3768 (6) log $\frac{81}{10,000}$ = log 3^4 − log 10^5 =
   4 log 3 − 5 = 4 × 0.4771 − 5 = −3.0916 (7) log (3 × 5 × 100) = log 3 + log 5 + 2 =
   0.4771 + 0.6990 + 2 = 3.1761 (8) log $\frac{25}{3}$ = 2 log 5 − log 3 = 0.9209