VANCOUVER COMMUNITY COLLEGE

Rational Functions:

Asymptotes

Consider a rational function, $f(x)$, in lowest terms of the form:

$$f(x) = \frac{P(x)}{Q(x)}$$

where the degree of $Q(x)$ is greater than or equal to 1. Then the asymptotes of $f(x)$ can be defined as follows:

Vertical Asymptote

A vertical asymptote occurs at each root of $Q(x) = 0$. The equation of the asymptotes are in the form x = a, where a is a root of $Q(x) = 0$.

Horizontal Asymptote

If the degree of $P(x)$ is less than the degree of $Q(x)$, then there is a horizontal asymptote at y = 0 (the x-axis).

If the degree of $P(x)$ is equal to the degree of $Q(x)$, then there is a horizontal asymptote at $y = \frac{p}{q}$, where p and q are the leading coefficients of $P(x)$ and $Q(x)$, respectively.

Oblique Asymptotes/ Slant Asymptotes and Others

If the degree of $P(x)$ is one greater than the degree of $Q(x)$, then there is an oblique or slant asymptote. If the degree of $P(x)$ is higher than the degree of $Q(x)$ by 2 or more, then the asymptote will be a curve. (This is rarely covered, and you are probably not responsible for it in your course.)

To find out the equation of the asymptote, perform the polynomial division $P(x) \div Q(x)$ and write the result as "quotient + $\frac{\text{remainder}}{\text{divisor}}$". The quotient is the equation of the asymptote.

The graph of a rational function *can* cross a non-vertical asymptote, but it *cannot* cross a vertical asymptote. (Non-vertical asymptotes only describe what a function does as it goes to ±∞, so for values somewhat close to 0, a function's graph can cross the asymptote.)

Note that a rational function has as many vertical asymptotes as its denominator has roots. (Double roots don't count twice, however.)

Notice, too, that a function can only have a horizontal asymptote or an oblique/slant asymptote, but not both (since the degree of $P(x) \le$ degree of $Q(x)$, or the degree of $P(x) >$ degree of $Q(x)$, but never both). If you find one of these types of asymptote, there

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is no need to waste time looking for the one.

EXERCISES

A. Determine the equations of the asymptotes of the following rational functions (You may assume anything big and ugly is not factorable):

1) $y = \frac{x-7}{x-2}$

4) $y = \frac{3x^2+5x-9}{x^2+2x-8}$

2) $y = \frac{x^3}{x^2-9}$

5) $y = \frac{4x^5-2x^4+3x^3-17x+19}{x^4-16}$

3) $y = \frac{4x^3+15x^2-7x+23}{(x-3)(x+5)(x-7)}$

6) $y = \frac{3}{x^2+16x+63}$

B. Write rational functions that have the following asymptotes and other features:

1) vertical asymptote at x = 5; horizontal asymptote at y = −3; passes through (0, 0)

2) vertical asymptotes at x = 3 and x = 4; horizontal asymptote at y = 0; numerator is a constant; passes through (1, 7)

3) vertical asymptote at x = −1 only; horizontal asymptote at y = 2; numerator is a quadratic; passes through (2, 0)

4) *Optional:* vertical asymptote at x = 3; oblique asymptote at y = x; passes through (6, 7)

SOLUTIONS

A. (1) x = 2; y = 1 (2) x = 3; x = −3; y = x (3) x = 3; x = −5; x = 7 ; y = 4
(4) x = −4 ; x = 2; y = 3 (5) x = −2; x = 2; y = 4x − 2 (6) x = −9; x = −7; y = 0

B. Answers will vary; these are probably the simplest. (1) $f(x) = \frac{-3x}{x-5}$ (2) $f(x) = \frac{42}{x^2-7x+12}$
(3) $f(x) = \frac{2x^2-8}{(x+1)^2}$ (4) $f(x) = \frac{x^2-3x+3}{x-3}$