Symmetry & Odd/Even Functions

It is useful to be able to tell whether the graph of a function has symmetry before we plot it. This saves us work when we do graph the function. We commonly look for **reflectional symmetry**, where flipping the graph around an axis does not change the graph, and **rotational symmetry**, where rotating a graph around a point does not change the graph.

CHECKING FOR SYMMETRY

If a function is defined in terms of x and y:

1) The function is **symmetric with respect to the x-axis** if, when you replace **y** with **−y** and simplify, you get the same function you started with.

2) The function is **symmetric with respect to the y-axis** if, when you replace **x** with **−x** and simplify, you get the same function you started with.

3) The function has **rotational symmetry with respect to origin** if, when you replace **y** with **−y** and **x** with **−x**, and simplify, you get the same function you started with.

4) The function is **symmetric with respect to the line y = x** if, when you exchange **y** and **x** and simplify, you get the same function you started with.

Example 1: Test the relation $2y = x^2 + 3$ for symmetry about both the coordinate axes, the origin and the line $y = x$.

Solution: We perform the four tests above on this equation and see what happens.

1) $2y = x^2 + 3$	2) $2y = x^2 + 3$	3) $2y = x^2 + 3$	4) $2y = x^2 + 3$
$2[-y] = x^2 + 3$	$2y = [-x]^2 + 3$	$2[-y] = [-x]^2 + 3$	$2[x] = [y]^2 + 3$
$-2y = x^2 + 3$ ✗	$2y = x^2 + 3$ ✓	$-2y = x^2 + 3$ ✗	$2x = y^2 + 3$ ✗

Only the second test worked, so this function is symmetric with respect to the y-axis only.

ODD AND EVEN FUNCTIONS

A function is an **even function** if $f(x) = f(-x)$ for all values of x in the domain of f. In other words, even functions are symmetric across the y-axis. In the graphs of even functions, if the point (x, y) is on the graph, then the point (−x, y) is too. If a polynomial function contains only even-numbered exponents and constant terms (and no absolute value signs), then it must be an even function.
A function is an **odd function** if $f(-x) = -f(x)$ for all values of x in the domain of f. In other words, even functions are rotationally symmetric around the origin. In the graphs of odd functions, if the point (x, y) is on the graph, then the point (−x, −y) is too. If a polynomial function contains only odd-numbered exponents (and no constant terms or absolute value signs), then it must be an odd function.

Example 2: Determine whether the function $f(x) = x^3 - x$ is even, odd, both, or neither.

Solution: We perform the tests for symmetry for even and odd functions:

Authored by Gordon Wong

$f(x) = x^3 - x$

$f(-x) = [-x]^3 - [-x] = -x^3 + x$

These two expressions are not the same, so this function is not even. Now we compare the result for $f(-x)$ to $-f(x)$:

$-f(x) = -(x^3 - x) = -x^3 + x = f(-x)$

These two expressions are the same, so the function is odd.

EXERCISES

A. Test the following relations for symmetry about the coordinate axes, the origin and the line y = x:

1) $y = x^4$
2) $y = x^3$
3) $y^2 = x^3$
4) $y = -x + 5$
5) $y = x^2 - 2$
6) $y = x^{-3}$
7) $y = |x|$
8) $x^3 = y^4 + 16$
9) $x = -4$
10) $y = 4$
11) $x^2 + y^2 = 1$
12) $y = 0$
13) $x = 5 - 3y$
14) $y = x$

B. Determine whether the following functions are even, odd, both or neither:

1) $f(x) = -x^2 + 16$
2) $f(x) = -x^5$
3) $f(x) = x^3 + 1$
4) $f(x) = x^{-2}$
5) $f(x) = x^3 + x$
10) $f(x) = |x| + 2$
11) $f(x) = |x|^3$
12) $f(x) = 7/x$
13) $f(x) = \dfrac{1}{x^2 + 1}$
14) $f(x) = x(x^2 + 2)$

6) $f(x) = -3x - 7$

7) $f(x) = 0$

8) $f(x) = 6x^4 + 7x^2$

9) $f(x) = 2x^3 - 5x$

15) $f(x) = x(x^3 + 2x)$

16) $f(x) = {}^{x^2}\!/_{2} + |x|^3$

17) $f(x) = \dfrac{|x|}{x^2 + 1}$

18) $f(x) = \dfrac{|x|}{x^3 - 1}$

C. One of the functions in part B was both odd and even. Explain why a function that passes through the point (1, 1) cannot be both odd and even. [*Hint: consider the descriptions of odd and even functions on this worksheet that talk about the point (1, 1).*]

SOLUTIONS

A. Symmetric around the… (1) y-axis (2) origin (3) x-axis (4) y = x (5) y-axis (6) origin (7) y-axis (8) x-axis (9) x-axis (10) y-axis (11) x-axis, y-axis, origin and y = x (12) x-axis, y-axis, and origin (13) none of them (14) origin and y = x

B. (1) even (2) odd (3) neither (4) even (5) odd (6) neither (7) both (8) even (9) odd (10) even (11) even (12) odd (13) even (14) odd (15) even (16) even (17) even (18) neither

C. If the function is even, then since (1, 1) is a point on the curve, (−1, 1) must also be a point on the curve. If the function were also odd, then since (1, 1) is a point on the curve, (−1, −1) would also be a point on the curve. If both (−1, 1) and (−1, −1) were both valid points for a relation, then the relation would fail the Vertical Line Test; if a vertical line is drawn through the graph of the relation, it would cross the graph at both these points. This is a contradiction, since that means the relation isn't a function in the first place.