VANCOUVER COMMUNITY COLLEGE

Conic Sections:

The Hyperbola

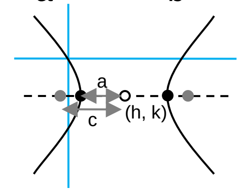
The equation for a hyperbola has both an x^2 and y^2 term, with one of them being added and the other subtracted. Once the equation is in standard form, which one is subtracted (x^2 or y^2) determines whether the hyperbola is "horizontal" or "vertical".

FORMULA FOR HYPERBOLAS

Once the formula for the hyperbola is in standard form (described below), *a* is always in the denominator of the term that's added, and *b* is always in the denominator of the term that's subtracted.

Horizontal Transverse Axis:

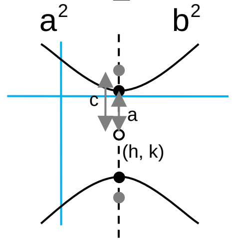
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$



centre: (h, k)
vertices: (h + a, k), (h − a, k)
foci: (h + c, k), (h − c, k), where $c^2 = a^2 + b^2$
asymptotes: $y - k = \frac{\pm b}{a}(x - h)$

Vertical Transverse Axis:

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$



centre: (h, k)
vertices: (h, k + a), (h, k − a)
foci: (h, k + c), (h, k − c), where $c^2 = a^2 + b^2$
asymptotes: $y - k = \frac{\pm a}{b}(x - h)$

Example 1: Find the centre, vertices, foci and asymptotes of the hyperbola $x^2 + 8x - y^2 + 10y = 13$.

Solution: First we need to get the equation into the standard form. We start by completing the squares for x and for y.

$$\begin{aligned}
(x^2 + 8x) - (y^2 - 10y) &= 13 \\
(x^2 + 8x + 16 - 16) - (y^2 - 10y + 25 - 25) &= 13 \\
(x^2 + 8x + 16) - 16 - (y^2 - 10y + 25) + 25 &= 13 \\
(x^2 + 8x + 16) - (y^2 - 10y + 25) &= 13 + 16 - 25 \\
(x + 4)^2 - (y - 5)^2 &= 4 \\
\frac{(x+4)^2}{4} - \frac{(y-5)^2}{4} &= 1
\end{aligned}$$

Now we can see h, k, a and b: h = −4, k = 5, a = 2 and b = 2. The x term is added, so its

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denominator has a^2. This hyperbola has a horizontal transverse axis. The centre is at (h, k), or (−4, 5). The vertices are at (h ± a, k), or (−2, 5) and (−6, 5). We calculate c:

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 2^2 + 2^2 = 8 \\ \therefore c &= 2\sqrt{2} \end{aligned}$$

The foci, then, are at (h ± c, k), or $(-4 \pm 2\sqrt{2}, 5)$. The asymptotes are:

$$\begin{aligned} y - k &= \frac{\pm b}{a}(x - h) \\ y - 5 &= \pm 2/2\,[x - (-4)] \\ y - 5 &= \pm(x + 4) \\ y - 5 = x + 4 &\text{ or } y - 5 = -x - 4 \\ y = x + 9 &\quad\;\; y = -x + 1 \end{aligned}$$

Example 2: Find the equation of the hyperbola with vertices at (5, 0) and (−5, 0) and foci at (6, 0) and (−6, 0).

Solution: First, we must determine whether this hyperbola has a horizontal or a vertical transverse axis. The two vertices have the same y value, as do the foci, so we have a horizontal transverse axis. (Vertical transverse axes have the same x value for all four points.) The distance between the two vertices is equal to 2a:

$$\begin{aligned} 2a &= \sqrt{[5-(-5)]^2 + (0-0)^2} \\ &= \sqrt{10^2 + 0^2} = 10 \\ \therefore a &= 5 \end{aligned}$$

The coordinate (5, 0) is the one that's farther to the right, so it must be (h + a, k). This means k = 0, and h + a = 5. Since a = 5, h must be 0.

We can get b by calculating the distance between the centre and either focus, which is c:

$$\begin{aligned} c &= \sqrt{[6-0]^2 + [0-0]^2} \\ &= \sqrt{6^2 + 0^2} = 6 \\ c^2 &= a^2 + b^2 \\ b^2 &= c^2 - a^2 \\ &= [6]^2 - [5]^2 \\ &= 36 - 25 = 11 \\ \therefore b &= \sqrt{11} \end{aligned}$$

We have h, k, a and b, so we can get the standard form of the equation of the hyperbola:

$$\begin{aligned} \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} &= 1 \\ \frac{(x-0)^2}{5^2} - \frac{(y-0)^2}{\sqrt{11}^2} &= 1 \end{aligned}$$

$$\frac{x^2}{25} - \frac{y^2}{11} = 1$$

Example 3: Find the equation of the hyperbola with vertices at (4, −15) and (4, 1) and asymptotes at y = 2x − 15 and y = −2x − 1.

Solution: First, we must determine whether this hyperbola has a horizontal or a vertical transverse axis. The two vertices have the same x value, so we have a vertical transverse axis. The distance between the two vertices is equal to 2a:

$$\begin{aligned} 2a &= \sqrt{[4-4]^2 + [(-15)-1]^2} \\ &= \sqrt{0^2 + 16^2} = 16 \\ \therefore a &= 8 \end{aligned}$$

The vertex at (4, 1) is the one that's farther up, so it must be (h, k + a). This means h = 4, and k + a = 1. Since a = 8, h must be −7.

Since a and b are distances, the equation for the asymptote with a positive coefficient on x must be of the form $y - k = \frac{a}{b}(x - h)$. In fact, the coefficient on x must be $\frac{a}{b}$:

$$\begin{aligned} \tfrac{a}{b} &= 2 \\ \tfrac{8}{b} &= 2 \\ b &= 4 \end{aligned}$$

We have h, k, a and b, so we can get the standard form of the equation of the hyperbola:

$$\begin{aligned} \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} &= 1 \\ \frac{(y+7)^2}{8^2} - \frac{(x-4)^2}{4^2} &= 1 \\ \frac{(y+7)^2}{36} - \frac{(x-4)^2}{16} &= 1 \end{aligned}$$

EXERCISES

A. Find the centre, vertices, foci and asymptotes for each hyperbola:

1) $\frac{x^2}{4} - \frac{y^2}{9} = 1$
2) $\frac{(y-1)^2}{4} - \frac{(x+1)^2}{9} = 1$
3) $x^2 - y^2 = 9$
4) $4x^2 - 4y^2 = 1$
5) $-x^2 + y^2 + 16y = 17$
6) $x^2 + 4x - y^2 + 8y = 3$
7) $x^2 + 2x - 4y^2 + 4y - 1 = 0$

B. Find the equation of a hyperbola with the following features:

1) vertices: (3, 0), (−3, 0); foci: (4, 0), (−4, 0)
2) vertices: (−1, 1), (−1, −3); foci: (−1, 2), (−1, −4)

3) vertices: $(-4, 10)$, $(-4, -2)$; asymptotes: $y = 3x + 16$, $y = -3x - 8$

C. Graph the hyperbola from Exercise A6, including the asymptotes.

SOLUTIONS

A: (1) ctr.: (0, 0); vert.: (2, 0), (−2, 0); foci: $(\pm\sqrt{13}, 0)$; asym.: $y = \pm\frac{3}{2}x$

(2) ctr.: (−1, 1); vert.: (−1, −1), (−1, 3); foci: $(-1, 1 \pm \sqrt{13})$; asym.: $y = \frac{2}{3}x + \frac{5}{3}$, $y = -\frac{2}{3}x + \frac{1}{3}$

(3) ctr.: (0, 0); vert.: (−3, 0), (3, 0); foci: $(\pm 3\sqrt{2}, 0)$; asym.: $y = x$, $y = -x$

(4) ctr.: (0, 0); vert.: $(-\frac{1}{2}, 0)$, $(\frac{1}{2}, 0)$; foci: $(\pm\frac{\sqrt{2}}{2}, 0)$; asym.: $y = x$, $y = -x$

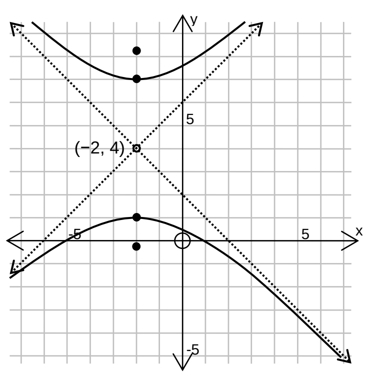
(5) ctr.: (0, −8); vert.: (0, −17), (0, 1); foci: $(0, -8 \pm 9\sqrt{2})$; asym.: $y = x + 8$, $y = -x + 8$

(6) ctr.: (−2, 4); vert.: (−2, 1), (−2, 7); foci: $(-2, 4 \pm 3\sqrt{2})$; asym.: $y = x + 6$, $y = -x + 2$

(7) ctr.: $(-1, \frac{1}{2})$; vert.: $(-2, \frac{1}{2})$, $(0, \frac{1}{2})$; foci: $(-1 \pm \frac{\sqrt{5}}{2}, \frac{1}{2})$; asym.: $y = \frac{1}{2}x + 1$, $y = -\frac{1}{2}x$

B: (1) $\frac{x^2}{9} - \frac{y^2}{7} = 1$ (2) $\frac{(y+1)^2}{4} - \frac{(x+1)^2}{5} = 1$ (3) $\frac{(y-4)^2}{36} - \frac{(x+4)^2}{4} = 1$

C:



[*Note: Even though the y² term was subtracted in this question, this hyperbola is vertical!*]