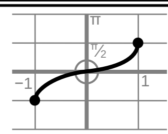
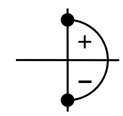
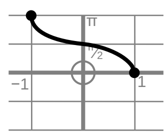
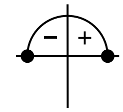
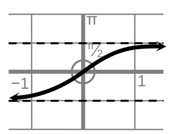
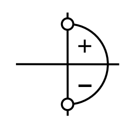
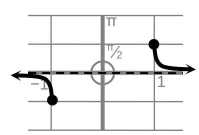
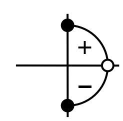
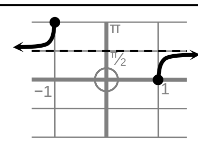
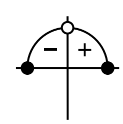
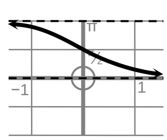
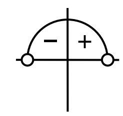
VANCOUVER COMMUNITY COLLEGE

Inverse Trigonometric Functions

Inverse trigonometric functions use the ratio of sides from the triangle to find an angle of the triangle. The inverse tan of 1 (or arctan 1) is:

$$\arctan 1 = \tan^{-1} 1 = 45° = \tfrac{\pi}{4}$$

Be careful! While $\sin^2 x$ is defined as $(\sin x)^2$, $\sin^{-1} x$ is not the same thing as $(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$. The "−1" in "$\sin^{-1} x$" is not an exponent (even though it looks like one). It's the symbol for an inverse function, as it is in "$f^{-1}(x)$". Any trig symbol with a "−1" is an inverse trig function, and it will give an angle as its final value.

FUNCTION	GRAPH	DOMAIN (x)	RANGE (θ)	
arcsine $y = \arcsin x = \sin^{-1} x$		$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	
arccosine $y = \arccos x = \cos^{-1} x$		$[-1, 1]$	$[0, \pi]$	
arctangent $y = \arctan x = \tan^{-1} x$		$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	
arccosecant $y = \text{arccsc}\, x = \csc^{-1} x$		$(-\infty, -1]$ $\cup [1, \infty)$	$[-\frac{\pi}{2}, 0)$ $\cup (0, \frac{\pi}{2}]$	
arcsecant $y = \text{arcsec}\, x = \sec^{-1} x$		$(-\infty, -1]$ $\cup [1, \infty)$	$[0, \frac{\pi}{2})$ $\cup (\frac{\pi}{2}, \pi]$	
arccotangent $y = \text{arccot}\, x = \cot^{-1} x$		$(-\infty, \infty)$	$(0, \pi)$	

To evaluate $\sec^{-1} x$, $\csc^{-1} x$, and $\cot^{-1} x$ on a calculator, *first* invert x. The easiest way to do this is to type [1] [÷] x. Then use the inverse function that is the reciprocal of the one in the question: sin for csc, cos for sec, and tan for cot.

Authored by Gordon Wong

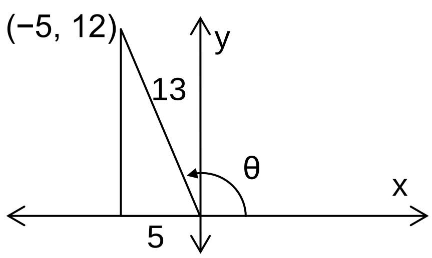
Example 1: Evaluate the following with exact values: (a) $\sin^{-1}(\sin \frac{7\pi}{4})$
(b) $\tan(\cos^{-1} -\frac{5}{13})$ (c) $\sin(\tan^{-1} 5)$

Solution: (a) A common mistake is to simply answer $\frac{7\pi}{4}$ since "sin" and "$\sin^{-1}$" are inverse operations. The domain and range of the inverse functions make this question not so straightforward.

As with normal order of operations, we'll start with the brackets and work our way out. $\sin \frac{7\pi}{4}$ is $-\frac{\sqrt{2}}{2}$, according to the special triangles. Now we have $\sin^{-1}(-\frac{\sqrt{2}}{2})$. The range of $\sin^{-1}$ does not include $\frac{7\pi}{4}$. It only goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, or quadrants I and IV. Because we're taking the inverse sine of a negative number, the angle must lie in quadrant IV.

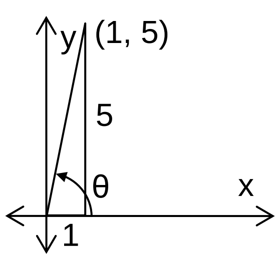
$\therefore \sin^{-1}(-\frac{\sqrt{2}}{2}) = -\frac{\pi}{4}$.

(b) Here, we have to create a triangle on the coordinate axes and use that to find the answer. We start from the inside. We're taking the arccosine of a negative number. The range of arccosine is in quadrants I and II, so this angle is in quadrant II. The ratio $-\frac{5}{13}$ tells us the lengths of the simplest possible triangle that could give us this result. Since $\cos = \frac{x}{r}$, the x-coordinate is −5 and r is 13. (It can't be x = 5 and r = −13; r is a distance, and can't be negative.) We draw a diagram:



Using the Pythagorean theorem, we can determine the length of the remaining side: it's 12. We could determine what θ is too, if we wanted, but it's actually not important. Now we find tan θ, whatever θ is. Since $\tan = \frac{y}{x}$, $\tan\theta = -\frac{12}{5}$.

(c) This question is similar to (b), but we don't have a fraction as a trig ratio. We can express the 5 as $\frac{5}{1}$. We draw another triangle. Since $\tan = \frac{y}{x}$, y = 5 and x = 1. This angle will be in quadrant I since everything is positive.



Again with the Pythagorean theorem we calculate that the hypotenuse of the triangle is $\sqrt{26}$. We need sin θ and since $\sin = \frac{y}{r}$, $\sin\theta = \frac{5}{\sqrt{26}} = \frac{5\sqrt{26}}{26}$.

EXERCISES

A. Evaluate without using a calculator. Express your answers in radians:

1) $\sin^{-1}\frac{1}{2}$
2) $\cos^{-1} 0$
3) $\tan^{-1}\frac{\sqrt{3}}{3}$
4) $\csc^{-1}\frac{1}{2}$
5) $\sec^{-1}(-1)$
6) $\cot^{-1}\frac{\sqrt{3}}{3}$
7) $\sin^{-1}\frac{\sqrt{2}}{2}$
8) $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$
9) $\sin^{-1} 0$
10) $\cos^{-1} 1$
11) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
12) $\tan^{-1}(-1)$
13) $\sin^{-1} 1$
14) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
15) $\cos^{-1}\frac{1}{2}$
16) $\tan^{-1}\sqrt{3}$

B. Evaluate using a calculator. Express your answer in degrees, rounded to two decimal places:

1) $\sin^{-1} 0.567$
2) $\sin^{-1}(-0.224)$
3) $\cos^{-1} 0.435$
4) $\cos^{-1}(-0.775)$
5) $\tan^{-1} 7.555$
6) $\tan^{-1}(-4.223)$
7) $\csc^{-1} 0.866$
8) $\csc^{-1}(-5.782)$
9) $\sec^{-1} 2.332$
10) $\sec^{-1}(-1.337)$
11) $\cot^{-1} 2.348$
12) $\cot^{-1}(-12.775)$
13) arcsin 0.538
14) arcsec (−2.864)
15) arccos (−0.954)
16) arccot 20.653
17) arctan 0.75
18) arccsc (−1.839)

C. Evaluate the following with exact values using right angle triangles and the Pythagorean Theorem:

1) $\cos(\sin^{-1} \frac{4}{5})$

2) $\sin(\tan^{-1} 2)$

3) $\tan(\sin^{-1} \frac{1}{4})$

4) $\tan(\cot^{-1} 6)$

5) $\sin[\cos^{-1}(-\frac{3}{5})]$

6) $\tan^{-1}(\tan \frac{3\pi}{4})$

7) $\tan[\sin^{-1}(-\frac{5}{13})]$

8) $\sin[\tan^{-1}(-3)]$

SOLUTIONS

A. (1) $\frac{\pi}{6}$ (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{6}$ (4) undefined (5) π (6) $\frac{\pi}{3}$ (7) $\frac{\pi}{4}$ (8) $-\frac{\pi}{4}$ (9) 0 (10) 0 (11) $-\frac{\pi}{3}$ (12) $-\frac{\pi}{4}$ (13) $\frac{\pi}{2}$ (14) $\frac{5\pi}{6}$ (15) $\frac{\pi}{3}$ (16) $\frac{\pi}{3}$

B. (1) 34.54° (2) −12.94° (3) 64.21° (4) 140.81° (5) 82.46° (6) −76.68° (7) undefined (8) −9.96° (9) 64.61° (10) 138.41° (11) 23.07° (12) −4.48° (13) 32.55° (14) 110.44° (15) 162.55° (16) 2.77° (17) 36.87° (18) −32.94

C. (1) $\frac{3}{5}$ (2) $\frac{2\sqrt{5}}{5}$ (3) $\frac{\sqrt{15}}{15}$ (4) $\frac{1}{6}$ (5) $\frac{4}{5}$ (6) $-\frac{\pi}{4}$ (7) $-\frac{5}{12}$ (8) $-\frac{3\sqrt{10}}{10}$