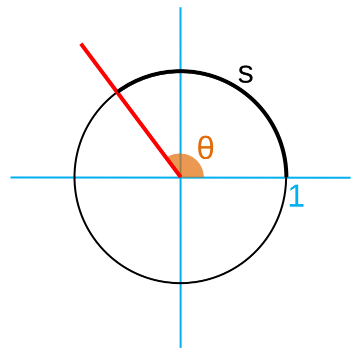
Radian Measure & Angular Mechanics

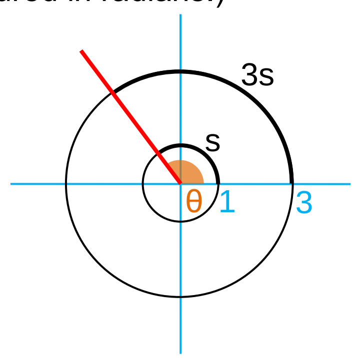
For some physics applications and for calculus, using degrees to measure angles is not convenient. Angles can also be expressed in **radian measure**, which is based on parts of the circumference.

RADIAN MEASURE



Consider a **unit circle**, a circle whose radius is 1, centred on origin. Angles are drawn counterclockwise from the positive x-axis. Any angle bigger than 360° goes around the circle more than once; any negative angle goes clockwise instead.

Any such angle contains some portion of the circumference within it. The angle can be defined in terms of this distance, the **arc length**, marked as s on the diagram. We can determine the length of s by looking at what fraction of the circle it represents. A 90° angle, for example, is ¼ of the way around the circle. The entire circumference is $C = 2\pi r = 2\pi$ and one-quarter of this is $2\pi \div 4 = \pi/2$, so "$\pi/2$ rad" is the radian measure of 90°. (Radian measures often mention pi, but they don't have to. They also often have the "rad" abbreviation after them, but again they don't have to. If it's clear that "1" is an angle, and there's no ° symbol, that angle is measured in radians.)



If we have a circle that is bigger or smaller, then all distances have been scaled by the same factor, but angles haven't. The circle at right has a radius of 3. In that case, the arc length described by θ is also 3 times as long, so we can get the same angle measure by dividing the arc length the radius. In other words,

$$\theta = \frac{s}{r}$$

Recall that we also use a ratio to gloss over the size of a shape when we take trigonometric ratios.

Example 1: Express the following angles in radians: a) 36° b) 75° c) 200°

Solution: We can use the equivalence between 360° and 2π rad to convert these angles. Radian measures are often written as fractions of π.

a) $36° \times \dfrac{2\pi \text{ rad}}{360°} = {}^{72\pi}/_{360} = {}^{\pi}/_{5}$ b) $75° \times \dfrac{2\pi \text{ rad}}{360°} = {}^{150\pi}/_{360} = {}^{5\pi}/_{12}$

c) $200° \times \dfrac{2\pi \text{ rad}}{360°} = {}^{400\pi}/_{360} = {}^{10\pi}/_{9}$

Angles and angular quantities are expressed with Greek letters. Distances, like radius and arc length, are expressed with letters from the English alphabet.

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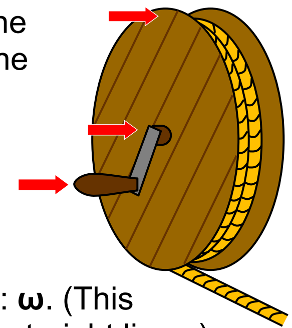
ANGULAR VELOCITY AND LINEAR VELOCITY

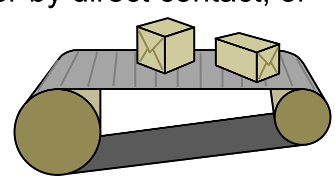


Three locations are marked on the winch in the diagram at the right: the end of the handle, the axle, and a point on the edge of the reel. Imagine turning the handle to wind the rope onto the reel.

In one sense, the entire device has the same speed: it spins at the same rate no matter which point we're discussing. All three points take the same length of time to make a complete rotation. This is the **angular velocity**. It's a measure of what total angle an object travels through in one unit of time. The variable for angular velocity is **omega**: **ω**. (This letter looks like a w, but it's always written with curves, never with four straight lines.) The proper unit for angular velocity is "radians per unit of time" such as ${}^{rad}/_{s}$ or ${}^{rad}/_{min}$. Spinning objects that share an axle are **coaxial**; coaxial objects always have the same angular velocity.

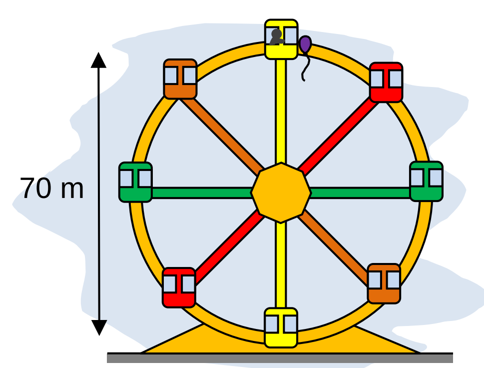
In another sense, some parts of the winch are travelling faster than others. The axle never moves, and without motion there's no speed. The point on the edge of the reel moves a long distance, the full circumference of the winch, with every rotation. The handle moves but not as far as the edge point. Here the radius makes a difference: the greater the radius, the more distance covered in the same time, so the higher the velocity. This is the more traditional way of describing velocity. For a spinning object, it's called **linear velocity**. The variable is v, and it's measured in the usual units for rates: ${}^{m}/_{s}$ or ${}^{km}/_{h}$ or ${}^{ft}/_{min}$…. Interpretations of linear velocity include:

- for a wheel, the linear velocity at the circumference is equal to the distance the wheel travels along the ground
- for the winch in the picture, the linear velocity at the distance from the axle to the rope is equal to the amount of rope that comes onto (or off of) the reel



If two spinning objects are connected at their circumferences, either by direct contact, or by a chain or rope, then their linear velocities must match. For the conveyor belt in the diagram, the two rollers must be moving with the same linear speed at their outside edges. If they weren't, that would mean that the belt was either getting longer or shorter, depending on which one is going faster. On the other hand, the rollers must have different angular velocities, since they have different radii.

We can convert between linear and angular velocities by dividing by r again: $\omega = \frac{v}{r}$.



Be careful about units! In a question you may be given a spinning object's speed that *looks* like a true angular velocity, but the units are wrong.

Example 2: At the PNE, a little boy lets go of a helium balloon while his father is riding the Ferris wheel, which is 70 m tall. The father wonders if he can catch the balloon as it goes by. The wheel is turning at 0.75 ${}^{rev}/_{min}$. If the balloon reaches the top of the wheel just as the father's car does, at what linear speed is he travelling at that moment?

Solution: First we need to convert the angular velocity to radian measure. One revolution is also equal to 2π radians:

$$0.75\ {}^{\text{rev}}\!/_{\text{min}} \times \frac{2\pi\ \text{rad}}{1\,\text{revolution}} = 1.5\pi\ {}^{\text{rad}}\!/_{\text{min}}$$

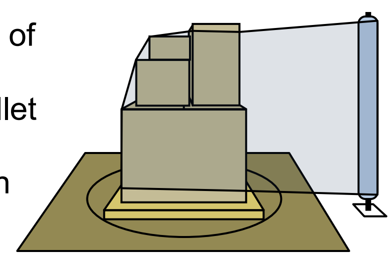
The diameter of the Ferris wheel is 70 m, so the radius is 35 m.

$$\begin{aligned} \omega &= \frac{v}{r} \\ v &= r\omega \\ &= (35\ \text{m})(1.5\pi\ {}^{\text{rad}}\!/_{\text{min}}) \\ &\approx 165\ {}^{\text{m}}\!/_{\text{min}} \end{aligned}$$

…or about 2.75 ${}^{\text{m}}\!/_{\text{s}}$, or just under 10 ${}^{\text{km}}\!/_{\text{h}}$, which might be a little fast for him.

In that final calculation that "rad" disappeared from the units. Radians aren't really a unit. We get radian measures from arc length divided by radius, measured in the same units; those units should cancel. We sometimes need a placeholder to indicate that, say, 1.5π ${}^{\text{rad}}\!/_{\text{min}}$ is an angular velocity, and describes something turning — it would be odd to write 1.5π$/_{\text{min}}$ — but when we don't need a placeholder anymore, it disappears.

Example 3: A factory's shipping department has a turntable. A pallet of boxes is placed on the centre and the end of a roll of plastic wrap is attached to them. When the turntable is started, the rotation of the pallet pulls the wrapping off of a roll. If the pallet has an average radius of 2 m, the turntable spins at 6 ${}^{\text{rev}}\!/_{\text{min}}$, and a roll is 15 cm in diameter, with what angular velocity will the roll spin when the turntable is started?



Solution: We should convert the rotation of the turntable into ${}^{\text{rad}}\!/_{\text{min}}$. None of these formulas work unless the angles are in radians.

$$6\ {}^{\text{rev}}\!/_{\text{min}} \times \frac{2\pi\ \text{rad}}{1\,\text{revolution}} = 12\pi\ {}^{\text{rad}}\!/_{\text{min}}$$

From there we can calculate the linear velocity of the pallet, and that value should be the same as the plastic wrap coming off the roll.

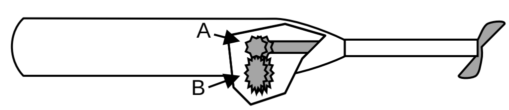
$$12\pi\ {}^{\text{rad}}\!/_{\text{min}} \times 2\ \text{m radius} = 24\pi\ {}^{\text{m}}\!/_{\text{min}}$$

We know the roll's diameter, so we can use the radius to get the roll's angular velocity.

$$24\pi\ {}^{\text{m}}\!/_{\text{min}} \div 0.075\ \text{m radius} = 320\pi\ {}^{\text{rad}}\!/_{\text{min}}$$

EXERCISES

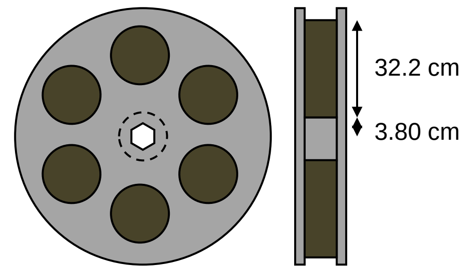
A. Inside an electric hand mixer, there is a system of small gears. One gear (A) drives the mixer's large blades because the blades are connected to an extension of the gear's axle. There is another slightly larger gear (B) meshed with gear A as well.



1) Do gear A and the mixer blades have the same linear velocity when they spin? Do they have the same angular velocity?

2) Do gears A and B have the same linear velocity when they spin? Do they have the same angular velocity?

B. A gear in an old film projector moves a film past the lens at a steady rate. Film displays 24 frames per second, and each frame is 1.86 cm long in the direction of the film. The film is currently being fed off a reel with the radius of 3.80 cm, and the film is 32.2 cm thick on the reel.



1) What is the linear velocity of the film passing by the lens? (This value is constant throughout the film.)

2) What is the current linear velocity of the film coming off the reel?

3) What is the current angular velocity of the film reel?

4) What will be the reel's final angular velocity when the film is about to run out?

5) What will be the final linear velocity of the film reel at that time?

6) Some of the answers for questions (1)–(5) change and some don't. For the ones that change, why do they change?

C. A man drives his motorcycle down a roadway at a constant speed. His tires are 86.0 cm in diameter.

1) His tires are rotating at 5.2 rev/s as he drives. What is ω? (It's going to come out as a decimal no matter what, so you don't need to express it in terms of π.)

2) How fast is he going in m/s? In km/h?

3) If it takes him 7.1 s to pass a fire hydrant, how far away was the hydrant?

4) The tires have a valve where it can be inflated 8 cm inward from the edge of the tire. Would you expect its linear velocity to be higher, lower, or the same as, you answer to (2)?

5) Would you expect the angular velocity of the valve to be higher, lower, or the same as, your answer to (1)?

6) What are the angular and linear velocities of the valve?

SOLUTIONS

A: (1) Because both objects are coaxial (like the winch and its handle) they have the same angular velocity. They're different sizes, so they do not have the same linear velocity. (2) Because the two gears' teeth go by at the same rate where they are meshed, the gears have the same linear velocity. The gears' radii are different, so they have different angular velocities.

B: (1) 44.6 cm/s (2) 44.6 cm/s (3) 1.24 rad/s (4) 11.7 rad/s (5) 44.6 cm/s

(6) It's because the radius changes. The linear velocity must remain constant — the answer to (1) should be the linear velocity throughout this problem since the movie shouldn't speed up. The angular velocity changes because the effective radius of the reel changes as film comes off the reel.

C: (1) 32.67 rad/s (2) 14.05 m/s = 50.57 km/h (3) 99.75 m (4) lower (5) the same

(6) 32.67 rad/s, new radius = 0.35 m, v = 11.44 m/s