Sequences & Series

Arithmetic sequences are lists of numbers, where each number is derived by adding to or subtracting from the previous number in the list, such as 1, 4, 7, 10, 13…. The number we add is called the **common difference (d)**. (If we subtract the same number every time, then d is negative.) The nth term in an arithmetic sequence is denoted by a_n. The sum of the terms in a sequence is called an **arithmetic series**. The symbol S_n represents the sum of the first n terms in a sequence.

For arithmetic sequences and series:

$$a_n = a_1 + (n - 1)d$$
$$S_n = \tfrac{n}{2}(a_1 + a_n) = \tfrac{n}{2}[2a_1 + (n - 1)d]$$

Example 1: Given an arithmetic sequence where $a_1 = 4$ and $d = 9$, determine a_6 and S_6.

Solution: If we're finding a_6 and S_6, our value for n for the sequence is 6. In the calculations below, n is plugged in separately so you can see all the places it goes. In your assignments this is not necessary.

$$\begin{aligned} a_{[6]} &= a_1 + ([6] - 1)d \\ &= [4] + 5[9] \\ &= 49 \end{aligned} \qquad \begin{aligned} S_{[6]} &= {}^{[6]}\!/_2[2a_1 + ([6] - 1)d] \\ &= 3(2[4] + 5[9]) \\ &= 159 \end{aligned}$$

Example 2: Given $a_4 = 16$ and $a_{10} = -14$, find d and S_{10}.

Solution: By the definition of an arithmetic sequence, we add d 6 times in getting from a_4 to a_{10}. So:

$$\begin{aligned} a_4 + 6d &= a_{10} \\ 16 + 6d &= -14 \\ 6d &= -30 \\ d &= -5 \end{aligned}$$

All our equations for S_n require a_1, so we must figure out its value:

$$\begin{aligned} a_n &= a_1 + (n - 1)d \\ a_4 &= a_1 + (4 - 1)d \\ 16 &= a_1 + 3(-5) \\ 31 &= a_1 \end{aligned}$$

Now we can calculate S_{10}:

$$\begin{aligned} S_n &= \tfrac{n}{2}(a_1 + a_n) \\ &= {}^{10}\!/_2[31 + (-14)] \\ &= 85 \end{aligned}$$

Authored by Gordon Wong

A list of numbers where each term is calculated by *multiplying* the previous term by some factor is a **geometric sequence**. That factor is the **common ratio (r)**. The sum of terms in a geometric sequence is a **geometric series**. It may be possible for a geometric series with an infinite number of terms to result in a finite total, if the terms get smaller as the series goes on. If the common ratio is between −1 and 1, exclusive, then the sum of such a series has a finite total. The sum of an infinite (geometric) series is written S_∞.

For geometric sequences and series:

$$a_n = a_1 r^{n-1}$$

$$S_n = \frac{a_1(1-r^n)}{1-r} = \frac{a_1 - r a_n}{1-r}, \text{ where } r \neq 1$$

$$S_\infty = \frac{a_1}{1-r}, \text{ where } |r| < 1$$

Example 1: Given a geometric sequence where $a_1 = 3$ and $r = 4$, find a_6 and S_6.

Solution:

$$a_6 = a_1 r^{n-1} = [3] \cdot [4]^{6-1} = 3072$$

$$S_6 = \frac{a_1(1-r^6)}{1-r} = \frac{[3](1-[4]^6)}{1-[4]} = 4095$$

Example 2: Given $a_3 = 64$ and $a_8 = -486$, find r and S_8.

Solution: By the definition of a geometric sequence, we multiply r 5 times in getting from a_3 to a_8. So:

$$\begin{aligned} a_3 \cdot r^5 &= a_8 \\ 64 \cdot r^5 &= -486 \\ r^5 &= -7.59375 \\ r &= \sqrt[5]{-7.59375} = -1.5 \end{aligned}$$

All our equations for S_n require a_1, so we must figure out its value first:

$$\begin{aligned} a_n &= a_1 r^{n-1} \\ a_3 &= a_1 r^{4-1} \\ 64 &= a_1(-1.5)^3 \\ 64 &= (3.375)a_1 \\ -18.\overline{962} &= a_1 \end{aligned}$$

Now we can calculate S_{10}:

$$\begin{aligned} S_n &= \frac{a_1(1-r^n)}{1-r} \\ &= \frac{-18.962 \cdot [1-(-1.5)^8]}{1-(-1.5)} \\ &= -33.\overline{185} \end{aligned}$$

Example 3: Insert (a) 3 arithmetic means between 29 and 57, and (b) 4 geometric means between 2 and 2048.

Solution: (a) Inserting arithmetic means (or geometric means) between two numbers means inserting numbers between them that will create a sequence. Here, we're making an arithmetic sequence by adding three numbers:

29, __, __, __, 57

We insert three numbers, but we take four steps to get to the second number. 29 is a_1 and 57 is a_5.

$$\begin{aligned} a_1 + 4d &= a_5 \\ 29 + 4d &= 57 \\ 4d &= 28 \\ d &= 7 \end{aligned}$$

We add 7 to 29 three times to get the arithmetic means 36, 43, 50.

(b) We will add four numbers to take the geometric sequence:

2, __, __, __, __, 2048

We insert four numbers and make five steps, so $2 = a_1$ and $2048 = a_6$.

$$\begin{aligned} a_1 \cdot r^5 &= a_6 \\ 2 \cdot r^5 &= 2048 \\ r^5 &= 1024 \\ r &= \sqrt[5]{1024} = 4 \end{aligned}$$

We multiply 2 by 4 four times to get the geometric means 8, 32, 128, 512.

Example 4: Express 0.82525252525…. as a fraction.

Solution: Repeating decimals can be thought of as infinite geometric series. First we group the decimal's digits, and convert them to fractions:

$$\begin{aligned} 0.8252525\ldots &= 0.8 + 0.025 + 0.00025 + 0.0000025 + \ldots \\ &= \tfrac{8}{10} + \tfrac{25}{1000} + \tfrac{25}{100{,}000} + \tfrac{25}{10{,}000{,}000} + \cdots \\ &= \tfrac{4}{5} + \tfrac{1}{40} + \tfrac{1}{4000} + \tfrac{1}{400{,}000} + \cdots \end{aligned}$$

The last three terms are the beginning of an infinite geometric series with $r = \frac{1}{100}$.

$$\begin{aligned} S_\infty &= \frac{a_1}{1-r} \\ &= \frac{\frac{1}{40}}{1-\frac{1}{100}} = \frac{\frac{1}{40}}{\frac{99}{100}} = \frac{1}{40} \div \frac{99}{100} = \frac{1 \times 100}{40 \times 99} = \frac{5}{198} \end{aligned}$$

So:

$$0.8252525\ldots = \frac{4}{5} + \frac{5}{198} = \frac{792}{990} + \frac{25}{990} = \frac{817}{990}$$

…and we can check this answer in a calculator by dividing.

EXERCISES

A. For each of these arithmetic sequences, (a) write the first three terms of the sequence, and (b) determine the requested information:

1) $a_1 = 2$, and $d = 4$; evaluate a_8 and S_8.
2) $a_{10} = 15$, $a_{14} = 7$; evaluate d and S_{14}.
3) $a_1 = 4$, $S_9 = 144$; evaluate a_9.
4) $a_1 = -3$, $a_8 = 11$; evaluate d and S_8.
5) $a_1 = 1$, $d = -3$, $S_n = -76$; evaluate n.

B. For each of these geometric sequences, (a) write the first three terms of the sequence, and (b) determine the requested information:

1) $a_1 = -2$, $r = 3$; evaluate a_5 and S_5.
2) $a_5 = 81$, $a_8 = -2187$; evaluate r and S_8.
3) $a_1 = 1$, $r = 2$, $a_n = 256$; evaluate n and S_n.
4) $a_1 = 2$, $r = \frac{1}{4}$; evaluate S_∞.
5) $a_1 = 20$, $S_\infty = 100$; evaluate r.

C. Given the sequence, find the indicated term or sum:

1) 4, 6.5, 9, . . .; find a_{86}.
2) 99, 93, 87, . . .; find S_{50}.
3) 0.2, 2, 20, . . .; find a_7.
4) 18, 6, 2, . . .; find S_∞.
5) the dates in March; find the sum of the series.

D. Insert the indicated means between the numbers provided:

1) 4 arithmetic means between −52 and 8
2) 3 geometric means between 5 and 405

E. Express these repeating decimals as fractions (improper fractions, where appropriate):

1) 0.7373737373…
2) 2.126126126…
3) 0.43333333…

SOLUTIONS

A. (1) 2, 6, 10; $a_8 = 30$, $S_8 = 128$ (2) 33, 31, 29; $d = -2$, $S_{14} = 280$ (3) 4, 7, 10; $a_9 = 28$ (4) −3, −1, 1; $d = 2$, $S_8 = 32$ (5) 1, −2, −5; $n = 8$

B. (1) −2, −6, −18; $a_5 = -162$, $S_5 = -242$ (2) 1, −3, 9; $r = -3$, $S_8 = -1640$ (3) 1, 2, 4; $n = 9$, $S_9 = 511$ (4) 2, $\frac{1}{2}$, $\frac{1}{8}$; $s_\infty = \frac{8}{3}$ (5) 20, 16, $\frac{64}{5}$; $r = \frac{4}{5}$

C. (1) 216.5 (2) −2400 (3) 200,000 (4) 27 (5) $S_{31} = 496$

D. (1) −40, −28, −16, −4 (2) 15, 45, 135 or −15, 45, −135

E. (1) $\frac{73}{99}$ (2) $\frac{236}{111}$ (3) $\frac{13}{30}$