Sigma Notation

A fast way to write the sum of a list of numbers that change in a predictable way is with **sigma notation**, also known as **summation notation**. Sigma notation uses a variable that counts upward to change the terms in the list. A typical sum written in sigma notation looks like this:

$$\sum_{k=0}^{4}(k^2+3)$$

The symbol "Σ" is the Greek capital letter sigma, which stands for "sum". The variable below it, k in this case, is called the **index of summation**, but you can think of it as a counter which increases one by one. The number at the bottom of sigma is the starting number for k (0) and the number at the top of the sigma is the last value of k (4). So k will take on all the integer values from 0 to 4, and each value of k will create one term in the sum. The terms are generated by plugging the values for k into the function following the sigma symbol. The last step is adding all the terms together.

Example 1: Find $\sum_{k=0}^{4}(k^2+3)$

Solution: $\sum_{k=0}^{4}(k^2+3) = ([0]^2 + 3) + ([1]^2 + 3) + ([2]^2 + 3) + ([3]^2 + 3) + ([4]^2 + 3)$

$= 3 + 4 + 7 + 12 + 19 = \underline{45}$

Example 2: How many terms are in the following sums:

a) $\sum_{n=1}^{20}(3n-1)$ b) $\sum_{i=0}^{10}(5i^3)$ c) $\sum_{t=5}^{12}\left(\frac{7920}{t}\right)$

Solution: The secret to counting terms in a sum is in the indices above and below the symbol, not in the function after it. There is one term for every value of the index of summation.

a) The numbers range from 1 to 20, so clearly there will be 20 terms.

b) The numbers range from 0 to 10, so it looks like there will be 10 terms, but this is a mistake. Going from 1 to 10, there would be ten terms, but i = 0 is an extra term so there are actually eleven terms. It is quite common for sums written in sigma notation to start counting from 0. Don't forget the zero term!

c) There's no rule that says you have to start counting from 1, or even 0. The numbers in this sum range from 5 to 12. The common sense solution is: these are the numbers from 1 to 12 without the numbers 1 to 4, so there are 12 − 4 = 8 terms. You can also use the following formula:

the number of terms in the sum $\sum_{k=a}^{b} f(k)$ is (b − a + 1)

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Example 3: Write the following sums in sigma notation.

a) x + 2x + 3x + 4x + 5x + … + 37x

b) 6 + 10 + 14 + 18 + 22 + 26 + 30

c) ½ − 1 + 1½ − 2 + 2½ − … + 16½

d) 10 + 40 + 90 + 160 + 250 + 360 + …

e) $\frac{1}{6} + \frac{2}{7} + \frac{3}{8} + \frac{4}{9} + \frac{5}{10} + \ldots + \frac{19}{24}$

Solution: The secret to writing sums in sigma notation is to figure out what's the same about all the terms, what's different about all the terms, and how to write what's different as increasing integers by using a counting variable, like k.

a) What's the same about the terms in the sum is that they all have an x. What's different is that consecutive integers from 1 to 37 have been multiplied by x to give terms of x to 37x. We represent the changing integers with a counting variable, like k, and write everything else that stays the same in the terms of the sum (in this case, x):

$$\sum_{k=1}^{37} kx$$

b) If we can't see what's the same, we can subtract every pair of consecutive terms to see what the differences are. For sum b, the difference between any two consecutive terms is always 4. This tells us that the counting variable has been multiplied by 4. In fact, the terms are all two more than a multiple of 4. If we "un-add" the twos and factor the terms to bring out the 4's, the consecutive integers can be written as:

$$(4 \times \mathbf{1} + 2) + (4 \times \mathbf{2} + 2) + (4 \times \mathbf{3} + 2) + (4 \times \mathbf{4} + 2) + \ldots + (4 \times \mathbf{7} + 2)$$

The integers range from 1 to 7, so:

$$\sum_{k=1}^{7} (4k + 2)$$

This isn't the only right answer. $\Sigma_{k=0}^{6}(4k+6)$ also works. (Do you see why?) An answer is correct as long as it produces the series described in the question.

c) This sum alternates between positive and negative terms. When this happens, ignore the signs (we'll deal with them later) and look at the rest of the sum:

$$\tfrac{1}{2} + 1 + 1\tfrac{1}{2} + 2 + 2\tfrac{1}{2} + \ldots + 16\tfrac{1}{2}$$

If we subtract consecutive terms from each other, the answer is always ½. So k has been multiplied by ½. We factor:

$$(\tfrac{1}{2} \times \mathbf{1}) + (\tfrac{1}{2} \times \mathbf{2}) + (\tfrac{1}{2} \times \mathbf{3}) + (\tfrac{1}{2} \times \mathbf{4}) + (\tfrac{1}{2} \times \mathbf{5}) + \ldots + (\tfrac{1}{2} \times \mathbf{33})$$

Now we deal with the alternating signs. We create alternating plus and minus signs by multiplying the function by either $(-1)^k$ or $(-1)^{k+1}$. We use the first one if the negative terms go with odd values for k, and we use the second one if the negative terms go with even values of k. Here, the term where k = 1 is positive and the term where k = 2 is negative, so we use the second form:

$$\sum_{k=1}^{33} \tfrac{1}{2}k \cdot (-1)^{k+1}$$

d) If we check the differences between consecutive terms, we get 30, 50, 70, … These numbers are different, so k hasn't been multiplied by one number. The differences are all divisible by 10, however, so we can factor out 10 from each term:

$$10 + 40 + 90 + 160 + 250 + 360 + \dots$$
$$= (10 \times \mathbf{1}) + (10 \times \mathbf{4}) + (10 \times \mathbf{9}) + (10 \times \mathbf{16}) + (10 \times \mathbf{25}) + (10 \times \mathbf{36}) + \dots$$

If we check differences between the bold numbers, we get 3, 5, 7, … The numbers increase by 2. When that happens, it means that the function involves k^2. Now we recognize the bold numbers:

$$= (10 \times \mathbf{1}) + (10 \times \mathbf{4}) + (10 \times \mathbf{9}) + (10 \times \mathbf{16}) + (10 \times \mathbf{25}) + (10 \times \mathbf{36}) + \dots$$
$$= (10 \times \mathbf{1}^2) + (10 \times \mathbf{2}^2) + (10 \times \mathbf{3}^2) + (10 \times \mathbf{4}^2) + (10 \times \mathbf{5}^2) + (10 \times \mathbf{6}^2) + \dots$$
$$= \sum_{k=1}^{\infty} 10k^2$$

Since the sum indicates that the list of terms never ends, we put the infinity symbol, ∞, on the top of the sigma symbol.

e) Here it's easy to see what's going on, but now we have two sets of changing numbers. The numerators count upwards, as we expect, and the denominators also count upwards, but not from the same starting place. We can "shift" the denominators by adding a number to k. The numbers 6, 7, 8… are what we get when we add 5 to the numbers 1, 2, 3…

$$\sum_{k=1}^{19} \frac{k}{k+5}$$

It might be tempting to use two different counting variables, but we don't do this in mathematics because it is always possible to change one of them to match the other.

Writing sums in sigma notation is one of those rare kinds of problems in math where there is no fixed method we can give you to follow. It's more creative and requires trial and error. Look for patterns. If you're not sure where to start, jump in and try something!

EXERCISES

A. Determine the number of terms in each of these sums.

1) $\sum_{k=1}^{61} (2k+1)$
2) $\sum_{k=1}^{9} \sqrt{k}$
3) $\sum_{k=0}^{30} k(x+1)$
4) $\sum_{k=0}^{5} k^2$
5) $\sum_{k=21}^{30} (k^2+k)$
6) $\sum_{k=1024}^{1039} k^2$

B. What are the first three terms of these sums?

1) $\sum_{k=1}^{5} (k+3)$
2) $\sum_{k=1}^{12} 4k^2$
3) $\sum_{k=0}^{6} x^k$
4) $\sum_{n=0}^{20} (2n+1)$
5) $\sum_{s=5}^{15} (s^2-s)$
6) $\sum_{i=4}^{10} \frac{120}{i}$

C. Evaluate these sums.

1) $\sum_{k=1}^{3} k$ 3) $\sum_{k=2}^{5} k(x-3)$ 5) $\sum_{k=0}^{5} 2^k$

2) $\sum_{n=1}^{6} \frac{n}{3}$ 4) $\sum_{m=13}^{20} 5$ 6) $\sum_{h=1}^{4} \frac{h}{h+1}$

D. Fill the brackets with the correct information to make the two sums equal. [*Hint: It may help you to write out the first few terms of the sum to see what it looks like.*]

1) $\sum_{k=1}^{6}(k+4) = \sum_{k=5}^{(\ \ )}\left(\qquad\qquad\right)$ 3) $\sum_{k=1}^{10}(k+7)^2 = \sum_{k=(\ \ )}^{(\ \ )} k^2$ 5) $\sum_{k=15}^{65}\sqrt{k} = \sum_{k=0}^{(\ \ )}\left(\qquad\qquad\right)$

2) $\sum_{g=1}^{9}\frac{1}{g+3} = \sum_{g=4}^{(\ \ )}\left(\qquad\qquad\right)$ 4) $\sum_{n=7}^{9}(n^5-1) = \sum_{n=1}^{(\ \ )}\left(\qquad\qquad\right)$ 6) $\sum_{k=4}^{13} x^k = \sum_{k=(\ \ )}^{(\ \ )} x^{k-7}$

E. Write these sums using sigma notation. (Many answers are possible.)

1) $2 + 4 + 6 + 8 + \ldots + 30$

2) $x^5 + x^6 + x^7 + x^8 + \ldots + x^{24}$

3) $1 + 6 + 11 + 16 + \ldots + 36$

4) $y^2 + y^5 + y^8 + y^{11} + \ldots + y^{26}$

5) $2 + 5 + 10 + 17 + 26 + 37 + 50 + 65$

6) $-\frac{1}{12} + \frac{1}{6} - \frac{1}{4} + \frac{1}{3} - \frac{5}{12} + \frac{1}{2} - \ldots + 1$

7) $\sqrt{3} + 2\sqrt{5} + 3\sqrt{7} + 4\sqrt{9} + 5\sqrt{11} + \ldots$

8) $24 + 12 + 8 + 6 + \frac{24}{5} + 4 + \frac{24}{7} + \ldots + 1$

SOLUTIONS

A. (1) 61 (2) 9 (3) 31 (4) 6 (5) 10 (6) 16

B. (1) 4 + 5 + 6 +… (2) 4 + 16 + 36 + … (3) $1 + x + x^2 + \ldots$ (4) 1 + 3 + 5 + …
(5) 20 + 30 + 42 + … (6) 30 + 24 + 20 + …

C. (1) 1 + 2 + 3 = 6 (2) $\frac{1}{3} + \frac{2}{3} + 1 + \ldots + 2 = 7$
(3) $(2x-6) + (3x-9) + (4x-12) + (5x-15) = 14x - 42$
(4) $5 + 5 + 5 + \ldots + 5 = 8 \times 5 = 40$ (5) 1 + 2 + 4 + 8 + 16 + 32 = 63
(6) $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} = \frac{163}{60} = 2.71667\ldots$

D. (1) $\sum_{k=5}^{10} k$ (2) $\sum_{g=4}^{12}\frac{1}{g}$ (3) $\sum_{k=8}^{17} k^2$ (4) $\sum_{n=1}^{3}[(n+6)^5-1]$ (5) $\sum_{k=0}^{50}\sqrt{k+15}$ (6) $\sum_{k=11}^{20} x^{k-7}$

E. (1) $\sum_{k=1}^{15} 2k$ (2) $\sum_{k=5}^{24} x^k$ (3) $\sum_{k=0}^{7}(5k+1)$ (4) $\sum_{k=1}^{9} y^{3k-1}$ (5) $\sum_{k=1}^{8}(k^2+1)$ (6) $\sum_{k=1}^{12}(-1)^k\left(\frac{k}{12}\right)$
(7) $\sum_{k=1}^{\infty} k\sqrt{2k+1}$ (8) $\sum_{k=1}^{24}\frac{24}{k}$