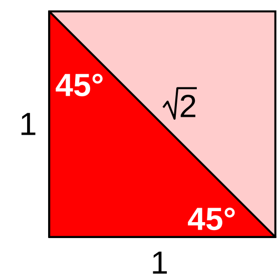
VANCOUVER COMMUNITY COLLEGE

The Special Triangles

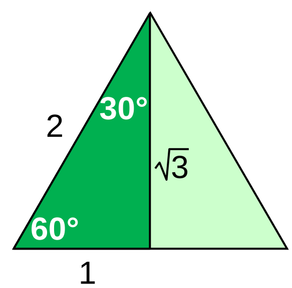
There are two ways to get precise answers for trig ratios of angles in right triangles. One is to work with a triangle based on a **Pythagorean triple** — a set of three whole numbers for which $a^2 + b^2 = c^2$ works, like 3, 4, 5 or 5, 12, 13. In these triangles the trig ratios work out easily, but the angles are messy.

The **special triangles** are two triangles that are very easy to work with in trigonometry. The angles are simple to calculate and their sides are easy to determine, so the trig ratios associated with them are easily expressed. You are expected to be familiar with this material, and using the triangles themselves as a mnemonic device can help you to remember it.

THE HALF-SQUARE AND THE HALF-EQUILATERAL TRIANGLE



If we start with a square and we cut it in half from corner to corner, we create a right triangle with two 45° angles. Since the size of a triangle doesn't matter in doing trig—only the ratios of the sides matter— we can decide that our original square has sides that are 1 unit long. Using the Pythagorean Theorem, we calculate that the hypotenuse is $\sqrt{2}$ units long.



If we start with an equilateral triangle and cut it in half from one corner to the centre of the opposite side, we create another right triangle. An equilateral triangle has three 60° angles. One of those is part of our right triangle, and the other one has been cut in half, so it's now 30°. The hypotenuse of the right triangle is one of the sides of the original equilateral triangle, and the shorter of the two legs is half of one of those sides. If we decide that the equilateral triangle's sides are 2 units, then the shorter leg is 1 unit long, and we can calculate that the other leg is $\sqrt{3}$ units long.

By remembering these particular triangles, and particularly the way they are constructed (by taking shapes and cutting them up), you can answer questions about the trig functions of these angles without needing to memorize a chart.

Example 1: Determine: (a) sin 45° (b) tan 30°

Solution: (a) The sine of an angle in a right triangle is the opposite over the hypotenuse. The 45° angle is in the half-square, and so the side opposite a 45° angle has length 1, and the hypotenuse has length $\sqrt{2}$. $\therefore \sin 45° = \frac{1}{\sqrt{2}}$.

(b) The tangent of an angle in a right triangle is the opposite over adjacent. We're working with a 30° angle, so we have to be careful about which side is the opposite and which one is the adjacent. The opposite side has length 1, and the adjacent side has length $\sqrt{3}$. $\therefore \tan 30° = \frac{1}{\sqrt{3}}$.

Authored by Darren Rigby

Example 2: A telephone pole is being replaced. The new pole is 4.6 m tall, and requires a guy wire attached to its top for added stability. If the guy wire must make a 60° angle with the ground, how much wire is needed?

Solution: The pole is opposite the 60° angle in the triangle, and the wire forms the hypotenuse, so we use the sine ratio. Call the unknown length of the wire x:

$$\sin 60° = \frac{\text{opp}}{\text{hyp}} = \frac{4.6}{x}$$

From the special triangle:

$$\frac{\sqrt{3}}{2} = \frac{4.6}{x}$$

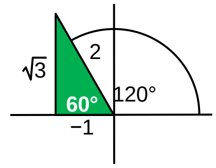
$$\sqrt{3}\,x = 2 \cdot 4.6$$

$$x = 5.311\ldots \text{ m}$$

RADIAN MEASURE AND THE UNIT CIRCLE

You can also use the special triangles to find reference angles within the unit circle. Because trig ratios eliminate size as a factor in calculations, it's okay that the "radius" for these triangles isn't 1.

Example 3: Determine: (a) cos 120° (b) cot 225°

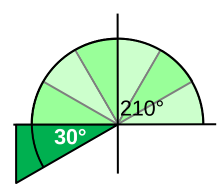


Solution: (a) 120° is in Quadrant II, so the reference angle is 180° − 120° = 60°. We can use the half-equilateral triangle as a reference triangle, as in the diagram. $\cos = \frac{\text{adj}}{\text{hyp}}$, so $\cos 120° = -\frac{1}{2}$.

(b) 225° is in Quadrant III, so the reference angle is 225° − 180° = 45°. $\cot = \frac{\text{adj}}{\text{opp}}$, so $\cot 225° = \frac{-1}{-1} = 1$.

You can also use the special triangles to remember the denominators for radian measures of angles. **4**5° is equivalent to $\frac{\pi}{\mathbf{4}}$, and any angle that uses 45° as a reference angle also has a denominator of 4. In the half-equilateral, they swap places: **3**0° is equal to $\frac{\pi}{\mathbf{6}}$ (and any angle with 30° as a reference angle also has a denominator of 6) and **6**0° is equal to $\frac{\pi}{\mathbf{3}}$ (and any angle with 60° as a reference angle also has a denominator of 3).

Example 4: Convert: (a) 210° to radians (b) $\frac{7\pi}{4}$ to degrees



Solution: (a) 210° is in Quadrant III, so the reference angle is 210° − 180° = 30°. We can use the half-equilateral triangle as a reference triangle, as in the diagram. With the 30° angle, we expect the denominator of the fraction that represents this angle to have the denominator 6 (from the **6**0° angle on the other side of the triangle). As the diagram shows, six-sixths of the semicircle gets us to 180°, and the angle we want is one more away. $210° = \frac{7\pi}{6}$ rad.

We can also think of ⅙ as 30°. Therefore $210° = 7 \times 30° = 7 \times \frac{\pi}{6} = \frac{7\pi}{6}$ rad.

(b) $\frac{7\pi}{4}$ has the denominator 4, which goes with the **4**5° angle in the half-square. If we visualize four quarters to make a semicircle, we need to go three quarters beyond that, which puts us in Quadrant IV. With a reference angle of 45°, that means we have an angle of 360° − 45° = 315°.

We can also think of ¼ as 45°. Therefore $\frac{7\pi}{4} = 7 \times \frac{\pi}{4} = 7 \times 45° = 315°$.

EXERCISES

A. Take a rectangle that measures 2 units by 1 unit. Cut the rectangle on the diagonal to produce a right triangle.

1) Determine the length of the hypotenuse of the "half-rectangle."
2) Find the measures of the acute angles in degrees and radians.
3) Why do you think the "half-rectangle" isn't considered a special triangle?

B. Determine:

1) $\cos 30°$
2) $\tan 45°$
3) $\sin 60°$
4) $\cos 45°$

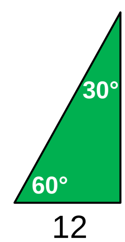
C. Determine:

1) $\csc 45°$
2) $\cos 135°$
3) $\tan 150°$
4) $\sin 300°$
5) $\sec 240°$
6) $\cot -120°$
7) $\cos 585°$
8) $\csc -390°$

D. Convert:

1) 30° to radians
2) 135° to radians
3) 300° to radians
4) −150° to radians
5) $\pi/3$ rad to degrees
6) $11\pi/6$ rad to degrees
7) $5\pi/4$ rad to degrees
8) $-2\pi/3$ rad to degrees

E. 1) Given the triangle to the right, determine the unknown side lengths:



2) A camera with a telescopic lens is on top of a building and pointing downwards at a 45° angle to focus on an intersection. If the height from the ground to the camera is 105 m, how far is it from the camera to the intersection?

3) A physics instructor demonstrates how using a pulley reduces the effort required to lift a block. He pulls the rope out at a 30° angle from the vertical of the pulley. If the block is level with the instructor's hand and there is 6 feet of rope from the block to the pulley, how long is the rope from the pulley to the instructor's hand?

SOLUTIONS

A: (1) $\sqrt{5}$ (2) 63.43° and 26.56°; 1.11 and 0.46 (or 0.35π and 0.15π)
(3) The half-rectangle is not a special triangle because even though the trig ratios would be simple to work with, the angles aren't. The special triangles have simple, common measures for the angles.

B: (1) $\frac{\sqrt{3}}{2}$ (2) 1 (3) $\frac{\sqrt{3}}{2}$ (4) $\frac{1}{\sqrt{2}}$

C: (1) $\sqrt{2}$ (2) $-\frac{1}{\sqrt{2}}$ (3) $-\frac{1}{\sqrt{3}}$ (4) $-\frac{\sqrt{3}}{2}$ (5) $-\frac{2}{\sqrt{3}}$ (6) $\sqrt{3}$ (7) $-\frac{1}{\sqrt{2}}$ (8) $-\frac{2}{\sqrt{3}}$

D: (1) $\pi/6$ (2) $3\pi/4$ (3) $5\pi/3$ (4) $-5\pi/6$ (5) 60° (6) 330° (7) 225° (8) −120°

E: (1) 24, $12\sqrt{3}$ (20.78) (2) $105\sqrt{2} \approx 148.5$ m (3) $4\sqrt{3} \approx 6.93$ ft