Essentials of Calculus 1

Limits

Some mathematical problems have solutions, but the standard way of solving them forces us to do “illegal” calculations, often involving infinity or indeterminate expressions. The tangent slope problem is an example of this: if the only point that we know about is one point of tangency, we can’t use the standard slope calculation of rise over run, since that requires two different points. If we plug the same point in twice we get 0/0.

We know that we cannot divide by zero, but not all division by zero is equally bad. If we pretend that the equation 2/0 = x has a solution, we could rearrange it to 2 = 0x. Here we see how bad it is — we know that no real number has this property. 2/0 is **undefined**: no real number has this value. On the other hand if we try the same analysis on 0/0 = x to get 0 = 0x, we see that we have the opposite problem! Now it’s so vague that literally any real number could be the solution. From just this information, we cannot determine which real number is the intended solution — the expression is **indeterminate**. So there’s hope for our slope problem. We just need another method of solution.

Let’s say that the curve increases to the right around the point of tangency. If we did the slope calculation on the point of tangency and a point just *slightly* to the right of it along the curve, we could find the slope of that secant line. The answer would be too steep to be right. We could also select an extra point just *slightly* to the left of the point of tangency and our slope would be too flat. The right answer is in between. If our definition of “slightly” shrinks, making these extra points a shorter and shorter distance from the real point of tangency, then our approximation of the answer improves.

In fact, we could look at the trend of what that slope formula does as the distance to the extra point decreases towards zero. As long as we don’t let it become zero, the formula always gives a real solution. This is the basis for the concept of the **limit**: the answer we get, when we extend the trend to *infinitesimally* small differences.

DERIVATIVES

The specific kind of limit — finding the slope of a tangent line by adapting the slope formula — is called the **difference quotient**:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Here, h is the distance between our extra point and the point of tangency whose x-coordinate is x in the difference quotient formula. If $f(x)$ is a polynomial, a rational expression of two polynomials, etc., then the way we expect the resolution of the difference quotient to go is like this: All the terms that do not include h in the expansion of $f(x + h)$ should cancel with the terms in $-f(x)$. After cancellation, all terms will have an h, which can be taken as a common factor and cancelled with the h in the difference

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quotient's denominator. At that point we can plug in $h = 0$ and resolve the limit.

Relatively quickly in Calc 1, we stop using the difference quotient for everything and start doing derivatives more directly, but derivatives are not the only application of limits — the introduction of the limit concept has many different consequences and applications. (Calc 2 has you exploring several, including integration, infinite series, and differential equations.)

GETTING TECHNICAL — THE PRECISE DEFINITION

You've probably heard that there was a historical controversy over whether it was Isaac Newton or Gottlieb Leibniz who invented calculus... but that wasn't the only controversy surrounding its creation. Newton, would symbolize small differences in numbers fluxions, with the letter o, taking care to note that this is not a digit zero. He would write equations like "$x = x + o$", treating the values as though they were too small to bother about:

"But further, since o is supposed to be infinitely small […], terms which have it as a factor will be equivalent to nothing in respect to the others. I therefore cast them out…"

—Sir Isaac Newton, Method of Fluxions

So the number o isn't zero, but it behaves like zero? Mathematicians at the time weren't going to stand for that! Some rebelled, refusing to believe that calculus was real math, even when it seemed to give correct, useful answers. It wasn't until later that the concept of a limit was developed, and the *precise definition of a limit* legitimized all the zero-but-not-zero sleight of hand.

The precise definition goes like this: If $\lim_{x\to a} f(x) = L$, then with the free choice of any small number ε (epsilon), it should always be possible to declare a small number δ (delta) so that if t belongs to the interval $(a - \delta, a + \delta)$, then $f(t)$ lies on the interval $(L - \varepsilon, L + \varepsilon)$. In other words, whatever tolerance in the value of L is called for, we must be able to establish a small neighbourhood around a so the function evaluated at any number in that neighbourhood will return an output within tolerance for L. The use of the Greek letters means this definition is sometimes called the **ε-δ definition of a limit**.

This definition reinforces the idea of a trend heading towards $f(a)$ whether or not $f(a)$ can be directly calculated. As we zoom in closer and closer to $x = a$ on a graph of $y = f(x)$, it should look more and more like $f(a) = L$, even if we never actually get to L. In fact, it's an important property of the limit that says that the function is never actually evaluated at that value a. Limits are only interested in the behaviour nearby.

Limits evaluated at finite values can be broken down into two one-sided limits: $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ (called the **left-hand limit** and **right-hand limit** respectively). We especially use these limits when we examine piecewise functions and absolute values — cases where a function behaves differently on each side of a. If the one-sided limits don't agree on an answer, the limit does not exist.

Around asymptotes, we can have $\lim_{x\to a} f(x) = \pm\infty$. Even if both one-sided limits tend to the same infinity (positive or negative), the limit still does not exist, even though we may write $\ldots = \infty$. (It fails to exist, but in a noteworthy way.)