Chain Rule

By now you should know how to take the derivatives of some standard functions, and how to take the derivatives of the products and quotients of functions. The **Chain Rule** allows us to take the derivative of composite functions. The derivatives of inside functions form a chain of factors in the derivative of the composite function.

Example 1: Find the derivative: $\sin^2 x$

Solution: $\sin^2 x = (\sin x)^2$. We want the derivative of something squared. We know:

$$f(v) = v^2$$
$$f'(v) = 2v$$

…but this is only true because we're taking the derivative with respect to v. In our question, we don't have a simple variable like v, we have sin x. We still take the derivative the same way, but there are two steps. First we take the derivative of the function "outside" the parenthesis by treating the inside part like a simple variable (e.g. just pretend everything inside is "x", which would make our function x^2 and the derivative of that 2x). Then we multiply by the derivative of the function <u>inside</u> the parenthesis (sin x), which would be cos x.

$$f(x) = \sin^2 x$$
$$f'(x) = 2(\sin x) \cdot \cos x$$
$$= 2 \sin x \cos x$$

Example 2: Find the derivative: $\sin(x^2)$

Solution: This is slightly different. This time x^2 is the one on the inside. The derivative of sin (THING) is cos (THING), but then we also have to multiply by the derivative of THING.

$$f(x) = \sin(x^2)$$
$$f'(x) = \cos x^2 \cdot 2x$$
$$= 2x \cos(x^2)$$

It is possible that we might have to use the Chain Rule more than once on a single problem to find the derivative.

Example 3: Find the derivative: $\cos^3(x^2 + x)$

Solution:

$$f(x) = \cos^3(x^2 + x)$$
$$= [\cos(x^2 + x)]^3$$
$$f'(x) = 3\cos^2(x^2 + x) \cdot [\text{derivative of } \cos(x^2 + x)]$$
$$= 3\cos^2(x^2 + x) \cdot [-\sin(x^2 + x)] \cdot [\text{derivative of } x^2 + x]$$
$$= 3\cos^2(x^2 + x) \cdot [-\sin(x^2 + x)] \cdot (2x + 1)$$
$$= -(6x + 3)[\cos^2(x^2 + x)][\sin(x^2 + x)]$$

Note: Do <u>*not*</u> actually write out "derivative of…" when showing work in your solutions! It is done here only as an aid to help you understand how the Chain Rule works.

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EXERCISES

A. Find the derivative:

1) $\cos(x^3)$ 3) $\sqrt{\tan x}$ 5) $e^{\sin x}$ 7) $\cos(\ln x)$

2) $\cos^3 x$ 4) $\tan\sqrt{x}$ 6) $\sin(e^x)$ 8) $\ln(\cos x)$

B. Find the derivative, and simplify, but do not expand fully:

1) $\sin(2x^2 - x + 3)$
2) $(\sqrt{x+4} + 3x)^2$
3) $\ln(x^3 + x - 8)$
4) $e^{(\sin x + x^2)}$
5) $\sin(\cos(\tan x))$
6) $(x^2 + 9)^7$
7) $\sqrt{x^2 + 4x - 7}$
8) $\sqrt[3]{\tan(x^3)}$

C. State whether the Chain Rule *needs* to be used, and find the derivative:

1) $\dfrac{x}{\cos x}$
2) $\dfrac{x}{\cos(x^2)}$
3) $x^3 \sin x$
4) $e^x \cdot \tan x + 1$
5) $(x + 5)^2$
6) $\ln(x^4 \cos x)$

D. Find the derivative (*Note:* ***really*** *hard!*). Simplify, but do not expand fully:

1) $\sin\left(\dfrac{x^3 + 4x + 5}{x - 2}\right)$
2) $\dfrac{x^9 \tan(\ln x)}{x - 5}$
3) $[\sin(x + \cos x)] \cdot [e^{(7x^2 - 3x + \tan 2x)}]$
4) $e^{\frac{x^3}{\cos x}} \cdot \ln\dfrac{\cos x}{x^3}$

SOLUTIONS

A. (1) $-\sin(x^3) \cdot 3x^2$ (2) $3\cos^2 x \cdot (-\sin x)$ (3) $\frac{1}{2}(\tan x)^{-1/2} \cdot \sec^2 x$ (4) $\sec^2\sqrt{x} \cdot \frac{1}{2}x^{-1/2}$
(5) $e^{\sin x} \cdot \cos x$ (6) $\cos(e^x) \cdot e^x$ (7) $-\sin(\ln x) \cdot \frac{1}{x}$ (8) $\frac{1}{\cos x} \cdot (-\sin x) = -\tan x$

B. (1) $(4x - 1)[\cos(2x^2 - x + 3)]$ (2) $(2\sqrt{x+4} + 6x)(\frac{1}{2\sqrt{x+4}} + 3)$ (3) $\frac{3x^2+1}{x^3+x-8}$
(4) $e^{(\sin x + x^2)} \cdot (\cos x + 2x)$ (5) $-\cos(\cos(\tan x))[\sin(\tan x)](\sec^2 x)$ (6) $14x(x^2 + 9)^6$
(7) $\frac{x+2}{\sqrt{x^2+4x-7}}$ (8) $\frac{x^2 \cdot \sec^2(x^3)}{\sqrt[3]{\tan^2(x^3)}}$

C. (1) no: $\sec x + x\frac{\sin x}{\cos^2 x}$ (2) yes: $\sec x^2 + 2x^2\frac{\sin(x^2)}{\cos^2(x^2)}$ (3) no: $3x^2 \sin x + x^3 \cos x$
(4) no: $e^x \tan x + e^x \sec^2 x$ (5) no, expand first: $2x + 10$ (6) yes: $\frac{4}{x} - \tan x$

D. (1) $\cos\left(\frac{x^3 + 4x + 5}{x - 2}\right) \cdot \left(\frac{2x^3 - 6x^2 - 13}{(x - 2)^2}\right)$ (2) $\frac{[9x^8 \tan(\ln x) + x^8 \sec^2(\ln x)] \cdot (x - 5) - x^9 \tan(\ln x)}{(x - 5)^2}$
(3) $\cos(x + \cos x) \cdot (1 - \sin x) \cdot e^{(7x^2 - 3x + \tan 2x)} + \sin(x + \cos x) \cdot e^{(7x^2 - 3x + \tan 2x)} \cdot (14x - 3 + 2\sec^2 2x)$
(4) $e^{\frac{x^3}{\cos x}} \cdot \frac{3x^2 \cos x + x^3 \sin x}{\cos^2 x} \cdot \ln\frac{\cos x}{x^3} - e^{\frac{x^3}{\cos x}} \cdot \frac{x \sin x + 3\cos x}{x \cos x}$