Curve Sketching

A good graphing calculator can show you the shape of a graph, but it doesn't always give you all the useful information about a function, such as its critical points and asymptotes. Due to most graphing calculators' poor resolution, it can also be difficult to get detailed information about the shape of a graph. Curve sketching is a kind of analysis that determines useful information about a function and allows you to draw a remarkably accurate graph. The example below illustrates all the steps of curve sketching.

Example: Sketch the graph of $f(x) = \dfrac{x^2}{4x+1}$.

DOMAIN AND RANGE

If the function has a limited domain or range, then this should be the first thing you examine. Domain is more important for curve sketching than range. Rational functions, radical functions, logarithmic functions, and some trigonometric functions can have limited domains. Some trigonometric functions have restricted ranges.

The function's denominator cannot be 0. There are no other restrictions on x.

$$\begin{aligned} 4x + 1 &\neq 0 \\ x &\neq -\tfrac{1}{4} \end{aligned}$$

The domain is $(-\infty, -\frac{1}{4}) \cup (-\frac{1}{4}, \infty)$.

INTERCEPTS

The x- and y-intercepts are the places where the function crosses the axes. These can be determined easily by evaluating the function at 0 (for x-intercepts) or solving the function for 0 (for y-intercepts). These points can be plotted directly onto the graph.

To find the y-intercept, we set x = 0:

$$\frac{[0]^2}{4[0]+1} = 0 \rightarrow (0, 0)$$

To find x-intercepts, we solve for $f(x) = 0$:

$$\begin{aligned} \frac{x^2}{4x+1} &= 0 \\ \therefore x^2 &= 0 \\ x &= 0 \end{aligned}$$

There is only one x-intercept. (There can never be more than one y-intercept; do you know why?)

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SYMMETRY
Functions can have reflectional, rotational and periodic symmetry. You can test for reflectional symmetry across the y-axis by testing whether $f(x) = f(-x)$ for all x in the domain. Such functions are called **even functions**. For functions that have rotational symmetry about the origin (called **odd functions**), $-f(x) = f(-x)$. **Periodic functions** repeat over time, such as the sine and cosine functions. If $f(x) = f(x + p)$ for some positive real number p and all x in the domain, the function is periodic.

We can tell by inspection that the function is not periodic. If it were, we'd have infinite x-intercepts at regular intervals, not just one at (0, 0). We should check for the other two forms of symmetry. Calculate $f(-x)$ and compare it to $f(x)$ and $-f(x)$.
$$f(-x) = \frac{(-x)^2}{4(-x)+1} = \frac{x^2}{-4x+1}$$
This is not equal to $f(x)$, so the function is not even. It is also not equal to $-f(x)$, which would have a denominator of $-4x - 1$, so the function is not odd.

ASYMPTOTES
Asymptotes are straight lines that the graph of a function approaches.

Vertical asymptotes occur at values of x where the function approaches infinity or negative infinity. For rational functions, these happen at values for x that make the function's denominator equal to zero. For other functions with restricted domains, it is worth testing the domain's endpoints to see if they are vertical asymptotes. The graph of a function will never cross a vertical asymptote.

Horizontal asymptotes occur at values of y where the function approaches infinity or negative infinity. If $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ come to fixed finite values, then these are the horizontal asymptotes. A function *may* take on the y value at a horizontal asymptote. Horizontal asymptotes are tendencies at positive or negative infinity for x.

To find horizontal asymptotes of rational functions of polynomials, look at the **order** of the polynomials in the numerator and denominator, i.e. the exponents on the leading terms. If the numerator has an order greater than that of the denominator, there is no asymptote. If the orders are the same, then the asymptote occurs at the value that you get when you divide the two leading terms. (So if the function is $3x^2 + \ldots$ over $5x^2 - \ldots$, there is a horizontal asymptote at $y = \frac{3}{5}$.) If the denominator has a higher order than the numerator, there is a horizontal asymptote at y = 0.

There is a vertical asymptote at the "hole" in our domain, $x = -\frac{1}{4}$. Because the order of the numerator is greater than the denominator (the numerator is quadratic while the denominator is linear), there is no horizontal asymptote.

CRITICAL VALUES

Calculate the first derivative of the function. Find all values for x where $f'(x) = 0$ and any endpoints of intervals where $f'(x)$ does not exist. These are the **critical values** for the function. At this stage, it is useful to set up a chart for yourself that looks like this:

x	
y	
y′	
y″	

For this chart, all values on the x line should be in order from smallest to largest, left to right. Put the critical values on the *x* line, and mark the *y′* line with zeroes (or DNE, for "does not exist") beneath them, to show that they are critical values. Plug these values for x into the function and determine the y-values. Put these on the *y* line.

To assess critical values, we look at the endpoints of the sections of our domain and anywhere that $f'(x) = 0$. We already know that there is a critical value at $x = -\frac{1}{4}$. Now we need to evaluate $f'(x)$, and solve for zero.

$$f(x) = \frac{x^2}{4x+1}$$

$$f'(x) = \frac{(2x)(4x+1)-(x^2)(4)}{(4x+1)^2}$$

$$= \frac{8x^2+2x-4x^2}{(4x+1)^2}$$

$$= \frac{4x^2+2x}{(4x+1)^2}$$

At critical values, $\frac{4x^2+2x}{(4x+1)^2} = 0$

$$\therefore 4x^2 + 2x = 0$$

$$2(x)(2x + 1) = 0$$

$$x = 0 \text{ or } x = -\tfrac{1}{2}$$

So there are three critical values: $-\frac{1}{2}$, $-\frac{1}{4}$ and 0. We set up our chart, including the y-values at the critical points:

x	−½	−¼	0
y	−¼	DNE	0
y′	0	DNE	0
y″			

INCREASING AND DECREASING INTERVALS

Choose test points between the critical values and plug them into y′. If a result is positive, then the interval between the two critical values is an increasing interval for the function. If a result is negative, the interval is a decreasing one. Mark the *y′* line with "+" and "−" to indicate increasing and decreasing sections.

We use test points between the critical values — any number we like — to see whether the four regions of the graph are increasing or decreasing. We evaluate $f'(x)$ for the test points:

$$f'(-1) = +\tfrac{2}{9};\ f'(-0.3) = -6;\ f'(-0.1) = -\tfrac{4}{9};\ f'(1) = +\tfrac{6}{25}$$

Positive results show increasing intervals and negative results show decreasing ones:

x	−1	−½	−0.3	−¼	−0.1	0	1
y		−¼		DNE		0	
y′	+	0	−	DNE	−	0	+
y″							

CONCAVITY AND INFLECTION POINTS

Calculate the second derivative. Find the inflection points of the function by solving $f''(x) = 0$. Mark these values in the appropriate places on the *x* line of your chart (still in order, smallest to largest, among the values already on the chart). Mark "0" at these places on the *y″* line of the chart, and evaluate the function for the inflection points. These values go on the *y* line; you should plot these points when you graph the function.

Determine which intervals of the function are concave up and concave down by evaluating $f''(x)$ at test values between each consecutive pair of inflection points. If $f''(x)$ is positive, the function is concave up, and if it's negative, the function is concave down.

We now perform a similar test with the second derivative. $-\frac{1}{4}$ is not technically an inflection point, but we will want to use it as a division point between concave-up and concave-down intervals.

$$f'(x) = \frac{4x^2 + 2x}{(4x+1)^2}$$

$$f''(x) = \frac{(8x+2)(4x+1)^2 - [2(4x+1)\times 4](4x^2+2x)}{(4x+1)^4}$$

We see a way to make this calculation easier by cancelling (4x + 1):

$$= \frac{(4x+1)\cdot[(8x+2)(4x+1) - 8(4x^2+2x)]}{(4x+1)^4}$$

$$= \frac{(8x+2)(4x+1) - 8(4x^2+2x)}{(4x+1)^3}$$

$$= \frac{32x^2 + 16x + 2 - 32x^2 - 16x}{(4x+1)^3}$$

$$= \frac{2}{(4x+1)^3} \neq 0$$

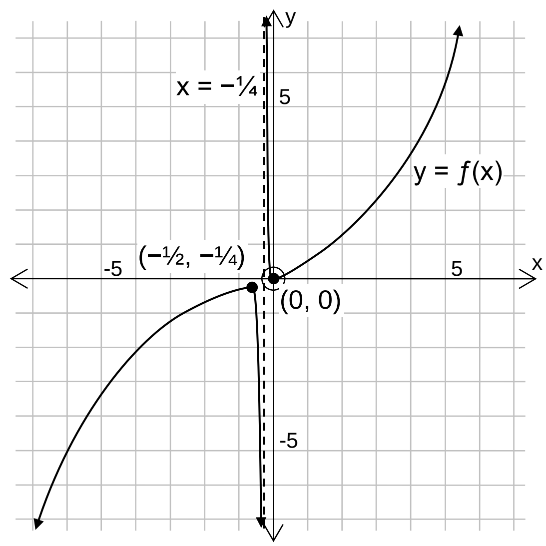
Inflection points occur where the second derivative equals zero. The only way to make a fraction equal to zero is if the numerator equals zero. Clearly this isn't possible in this case, so there are no inflection points on this curve. So we have two sections of the graph to check, one on either side of the discontinuity at x = $-\frac{1}{4}$.

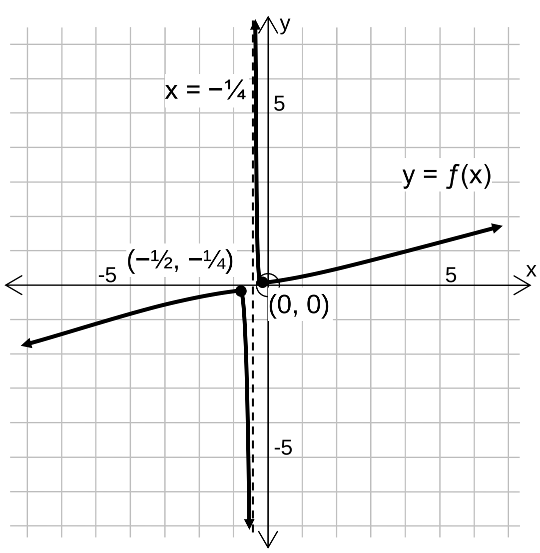
$$f''(-1) = -\tfrac{2}{27}\ ;\ f''(0) = +2$$

x	−1	−½	−0.3	−¼	−0.1	0	1
y		−¼		DNE		0	
y′	+	0	−	DNE	−	0	+
y″	−			DNE		+	

The interval below the discontinuity is concave down and the interval above the discontinuity is concave up.

We now have enough information to make a sketch of the graph. There is a vertical asymptote at x = $-\frac{1}{4}$. Below this discontinuity the function is concave-down with a maximum at $(-\frac{1}{2}, -\frac{1}{4})$. Above the discontinuity, the function is concave-up with a minimum at (0, 0), which is the graph's only intercept. The graph tends to negative infinity as x → −∞ and as x approaches $-\frac{1}{4}$ from below. The graph tends to infinity as x → ∞ and as x approaches $-\frac{1}{4}$ from above. Our sketch is to the right, and the actual graph is below.





The only thing our sketch didn't capture was how fast the two ends tended to infinity. (There was one more test we could have done that would have told us this: the test for a slant asymptote. Not every class covers it. If yours does, you may want to go back and calculate the slant asymptote for this function.)

EXERCISES

A. Using a graphing calculator, plot the graph of $y = \dfrac{x+6}{x^2+2x-15}$.

1) From the graph, estimate or calculate the locations of asymptotes and critical points for the function.

2) Use the techniques of curve sketching to verify your answer to (1).

B. Sketch the graphs of the following functions. Include intercepts, critical values, inflection points and asymptotes on the graph where applicable:

1) $f(x) = -\dfrac{3x+2}{5x-1}$

2) $f(x) = x^4 + 6x^3 + 4x^2$

3) $f(x) = \sin x + \frac{1}{2}\sin 2x$

SOLUTIONS

A. (1) Most likely answers: vertical asymptotes at $x = -5$ and $x = 3$, horizontal asymptote at $y = 0$, critical point at $(-3, -0.25)$. (2) Vertical asymptotes at -5 and 3; no horizontal asymptotes; critical points at $(-3, -\frac{1}{4})$ and $(-9, -\frac{1}{16})$

B. (1) Domain: $(-\infty, \frac{1}{5}) \cup (\frac{1}{5}, \infty)$; Intercepts: $(0, 2)$, $(\frac{2}{3}, 0)$; Symmetry: none; Asymptotes: $x = \frac{1}{5}$, $y = -\frac{3}{5}$; Critical value: $\frac{1}{5}$; Inflection point: none; increasing everywhere in domain; concave up on $(-\infty, \frac{1}{5})$, concave down on $(\frac{1}{5}, \infty)$

(2) Domain: $\mathbb{R}$; Intercept: $(0, 0)$, $(0, -3 \pm \sqrt{5})$; Symmetry: none; Asymptotes: none; Critical points: $(0, 0)$, $(-\frac{1}{2}, 0.3125)$, $(-4, -64)$; Inflection points: $x = \frac{-9 \pm \sqrt{57}}{6}$, so $(-0.24, 0.15)$, $(-2.76, -37.60)$; increasing on $(-4, -\frac{1}{2}) \cup (0, \infty)$ and decreasing on $(-\infty, -4) \cup (-\frac{1}{2}, 0)$; concave up on $(-\infty, -2.76) \cup (-0.24, \infty)$ and concave down on $(-2.76, -0.24)$

(3) Domain: $\mathbb{R}$; Intercepts: $(\pi \cdot p, 0)$ for all $p \in \mathbb{N}$; Symmetry: periodic (period = 2π), odd; Asymptotes: none; Critical points: $(\frac{\pi}{3}, \frac{3\sqrt{3}}{4})$, $(\pi, 0)$, and $(\frac{5\pi}{3}, -\frac{3\sqrt{3}}{4})$; Inflection points: $x = 0$, π and $\cos^{-1}(-0.25)$, so $(0, 0)$, $(1.82, 0.73)$, $(\pi, 0)$, $(4.46, -0.73)$; within $[0, 2\pi)$, increasing on $[0, \frac{\pi}{3}) \cup (\frac{5\pi}{3}, 2\pi)$ and decreasing on $(\frac{\pi}{3}, \frac{5\pi}{3})$; concave up on $(1.82, \pi) \cup (4.46, 2\pi)$ and concave down on $[0, 1.82) \cup (\pi, 4.46)$.