The Implications of

Conditional Statements

The implication operator in symbolic logic is one of the hardest to understand for a couple of reasons. First, implications translate to "if" statements in English, and we use the word "if" in a wide variety of ways, and second, implication statements are true under unusual circumstances when we translate them to English.

From a logical standpoint, an implication relates two statements: $p \rightarrow q$. The p here may be called the **premise**, the **hypothesis**, or the **supposition**, among other things. The q is called the **conclusion**. We can rewrite the statement $p \rightarrow q$ using simpler operators: $\neg p \vee q$. We can also look at the truth table for implications:

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

The only time an implication statement is false is if its supposition is true, but its conclusion is false. That's why we associate the $\rightarrow$ operator with the word "if". A typical "if" statement describes the logical progression of a sequence of events.

We need to be able to turn English sentences into logical propositions, so it's important that we understand how "if" is used in English, and what that means for that one false condition for the overall statement.

We use if statements to describe cause and effect.

If you rub the cat's fur the wrong way, she'll scratch you.

If you log in using the admin password, you can access the registry.

The clause before the comma in these cases is the supposition, and the clause after is the conclusion. Let's look at the cat sentence and examine its truth values.

Case 1) You rub the cat's fur the wrong way (p = 1). She scratches you (q = 1).
The outcome that I warned you about came true. The if-statement was true.

Case 2) You rub the cat's fur the wrong way (p = 1). She doesn't scratch you (q = 0).
This is the obvious case of the statement being false. The warning wasn't accurate.

Case 3) You ***don't*** rub the cat's fur the wrong way (p = 0). She ***doesn't*** scratch you (q = 0).
Good, you heeded the warning, and everything worked out. The warning was still true: if you rub her fur the wrong way later, she'll *still* scratch you.

Case 4) You ***don't*** rub the cat's fur the wrong way (p = 0). She ***does*** scratch you (q = 1).
Maybe she's just a mean cat, and she'll scratch you anyway. The warning never promised that she wouldn't. It only promised that *if* provoked, the cat will scratch. It didn't address the outcome if you didn't provoke the cat.

We use if statements to make wishes and promises.

If I ruled the universe, I'd give everyone a million dollars.

If I see him, I'll tell him you stopped by.

With this sort of if-statement the supposition is out of our control or isn't likely to occur.

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Let's look at the "million dollars" sentence and examine its truth values.

Case 1) I rule the universe (p = 1). I give everyone a million dollars (q = 1).
Again, the true-true case is easy. I did what I said I would, so the statement is true.
Case 2) I rule the universe (p = 1). I ***don't*** give everyone a million dollars (q = 0).
This was probably the reason you elected me ruler of the universe, and I didn't deliver on my campaign promise. Here, I lied; the if-statement is false.
Case 3) I don't rule the universe (p = 0). I ***do*** give everyone a million dollars (q = 1).
I don't *need* to rule the universe to be nice. Clearly I would follow through on what I said, so the statement is true.
Case 4) I don't rule the universe (p = 0). I ***don't*** give everyone a million dollars (q = 0).
This is the tough one to understand. I ***can't*** follow through on my wish, but the action that triggers me following through also hasn't happened, so I don't need to follow through yet. On the other hand, this is a legitimate expression of how I feel. This is considered a true statement (and it'll continue to be true until I become your leader and then stiff everyone).

We use if statements to speculate about "what might have been".
If I had studied last night, I'd have passed the test this morning.
If I were you, I'd stay away from that dog.
Here we know the supposition isn't really true. Let's look at the test sentence and examine its truth values.

Case 1) I studied last night (p = 1). I passed the test this morning (q = 1).
I believe that this is what would have happened. It's the sentiment of the statement. It's true.
Case 2) I studied last night (p = 1). I ***didn't*** pass the test this morning (q = 0).
My speculation was wrong. Even studying wouldn't have been enough to pass. The if statement is incorrect; it's false.
Case 3) I didn't study last night (p = 0). I ***did*** pass the test this morning (q = 1).
This *case* is unlikely to happen with this statement — why would I say it if I really did pass? Maybe I did better than I thought. Maybe I got bell-curved up to a D. The fact that I'm unlikely to say it doesn't affect whether the statement is a true one. I passed. Studying on top of that isn't going to mean I'd fail instead. The statement is true.
Case 4) I didn't study last night (p = 0). I didn't pass the test this morning (q = 0).
My statement says that studying would make the difference. We'll never prove it, but the statement is accurate as far as we know, and we have no way to know it's wrong. We consider it to be true.

As one final investigation, we can change all these if-statements into or-statements using the definition of the "→" symbol: $p \rightarrow q \coloneqq \neg p \vee q$

"If you rub the cat's fur the wrong way, she'll scratch you," can also be expressed as, "Don't rub the cat's fur the wrong way, or she'll scratch you."
"If I ruled the universe, I'd give everyone a million dollars," can be read as "It's too bad I don't rule the universe, or else I'd give everyone a million dollars."
"If I had studied last night, I'd have passed the test this morning," can be interpreted as, "I didn't study last night; otherwise I would have passed the test this morning."

All three of these new expressions are logically equivalent to the if-statements we started with, even though the nuances of the sentences are slightly different. This highlights one of the problems with interpreting English speech — even simple sentences are loaded with baggage. The way we're writing the "test" sentence implies an additional statement: I didn't study, $\neg p$. This is not, however, a strictly logical interpretation of the sentence itself, and that's what we're concerned with for this course.

EXERCISES

A. Consider the following primitive statements:

a: The printer is plugged in.
b: The network is down.
c: The printer is jammed.
d: The printer is working.
e: My job gets to the printer.

Write the following English sentences as compound propositions using the → symbol (and possibly others).

1) If the printer is working then it's plugged in.

2) If my job doesn't get to the printer, then either the network is down or the printer isn't plugged in.

3) If the printer is jammed then it's not working, and that means my job won't get to the printer.

B. Consider the following additional primitive statements:

p: I'm in the computer lab.
q: I can fix the printer.
r: My report gets printed.

Write English sentences for the following compound propositions:

1) $d \rightarrow \neg q$

2) $[p \wedge [(\neg a \vee c)] \rightarrow q$

3) $(b \rightarrow \neg e) \rightarrow \neg r$

4) $(q \wedge a \wedge \neg c) \rightarrow b$

C. The new operating system isn't ready yet. The graphics are done, but the interface has bugs. The system crashes if the user puts a slash in his password. We can't put any more programmers on it, or our other projects will suffer. We need $10,000 more or 5 more weeks to finish the project. The client won't be happy if the project is delayed more than a month, and they won't pay us in that case. I need a vacation.

Based on this information, determine whether these statements are true (1), false (0), or unknown (?).

1) If the graphics are done, then the operating system is ready.

2) If the system crashes for any reason then the interface has bugs.

3) If we put more programmers on it, then our other projects will suffer.

4) If the boss gives me any more work to do, then I need a vacation.
5) The user put a slash in his password.
6) If we don't get $10,000 more, the client won't pay us.
7) If the graphics aren't done, then the client is happy.

D. Rewrite the following as if-statements.
1) Eat your dinner or you won't get dessert.
2) Had I been there, I could have helped.
3) I'll be in the office, should anyone need me.
4) Reegan didn't get her parents' permission, or else she'd be going on the trip.
5) It's a shame I'm not faster. I could have caught him.
6) I would never have noticed that without you telling me about it.
7) We're going to the picnic tomorrow unless it's raining.
8) That snow hasn't covered these tracks tells us that the deer is nearby.

SOLUTIONS

A: (1) $d \rightarrow a$ (2) $\neg e \rightarrow (b \vee \neg a)$ (3) $(c \rightarrow \neg d) \rightarrow \neg e$ or $c \rightarrow (\neg d \wedge \neg e)$

B: *Other answers are possible:* (1) If the printer is working then I can't fix it.
(2) If I'm in the computer lab and the printer is unplugged or jammed then I can fix it.
(3) If the network is down, then my job won't get to the printer, so my report won't get printed.
(4) If I can fix the printer, and it's plugged in and not jammed, then the network must be down.

C: (1) $1 \rightarrow 0 = 0$ (2) $1 \rightarrow 1 = 1$ (3) 1: This is the if-statement version of the fourth sentence in the paragraph.
(4) $? \rightarrow 1 = 1$: A conditional statement is only false when $1 \rightarrow 0$. Knowing the conclusion is true is enough to know the overall statement is true.
(5) ?: We don't know if the supposition or conclusion of a given conditional statement is true without more information.
(6) 1: Logically, if we don't get $10,000, then we need 5 more weeks to finish the project. That will delay the project by more than a month. This implies that the client won't pay us.
(7) $0 \rightarrow ? = 1$: Never mind that it's nonsense! We can still evaluate the statement. The supposition is false, so the overall statement is true, by reasoning similar to that of Question C4.

D: *Other answers are possible:* (1) If you don't eat your dinner, you won't get dessert.
(2) If I had been there, I could have helped.
(3) If anyone needs me, I'll be in the office.
(4) If Reegan had her parent's permission, she'd be going on the trip.
(5) If I were faster, I could have caught him.
(6) If you hadn't told me about that, I would never have noticed it.
(7) If it's not raining, we're going to the picnic tomorrow.
(8) If snow hasn't covered these tracks, the deer is nearby.