VANCOUVER COMMUNITY COLLEGE

Set Theory:

Elements & Subsets

A set is a mathematical container. We can describe a set by saying exactly what's inside it, so that {monitor, printer, scanner, headphones} is a set — surrounded by curly brackets with a list inside — or we can describe it by giving a broad definition of what goes in the set, so that $\{x \mid x \text{ is a month of the year}\}$ is a set. Both of these examples are **well-defined** sets. If I suggested an item, you would be able to determine whether that item is a part of each set. (This does not necessarily mean it's possible to list every element of a well-defined set. The set $\{x \mid x \in \mathbb{R}, x < 5\}$ is well-defined, but the list itself is infinitely large. On the other hand, if I give you −2.1, 17, freedom and π, you should be able to tell which of these suggestions belong to that last set.) Listing the elements of a set in a different order doesn't change the set, nor does listing an element of a set more than once. The set {a, b, c} could be written {c, a, b} or {b, a, c, c, a, b, c} without changing the set.

As we did with quantified statements, if we define our set by definition, rather than by an exhaustive list, we need to specify a **universe** for the set. Given the set $\{x \mid x < 5\}$, I don't know whether π is in the set. The universe that the set applies to could be integers, and therefore π isn't in the correct universe.

Any item that appears in a set is called an **element** of the set. If we call the set of computer peripherals set C, then we can say scanner $\in C$ (read, "scanner is an element of C", or "scanner belongs to C", and blender $\notin C$ ("blender is not an element of C", "blender does not belong to C").

We can also compare sets. Two sets, A and B, are **equal** if every element of A is also in B, and every element not in A is also not in B. (So in the universe of integers, the sets {…−1, 2, 5, 8, …} and $\{x \mid x + 1 \text{ is divisible by } 3\}$ are equal, since they contain all the same elements, and neither has an element the other one lacks.)

One set, A, is a **subset** of another set, B, if all the elements of A are also elements of B. We write $A \subseteq B$. Call the set of vowels V, V = {a, e, i, o, u}, and the set of letters L, L = {a, b, c, …, x, y, z}. Then $V \subseteq L$. Besides that, the set L contains elements not found in V — b, c, d, etc. aren't in V — so V is a **proper subset** of L. It's a subset but it's also "smaller". We write $V \subset L$. (Another way of seeing the difference between "proper subset" and "subset": the difference between "$A \subseteq B$" and "$A \subset B$" is the same as the difference between "$x \leq 5$" and "$x < 5$" — the second expression does not allow the two sides to be equal.)

The **cardinality** of a set is the number of elements in the set. The cardinality of the set V is 5, written $|V| = 5$. Also, $|L| = 26$, and the cardinality of the set {b, a, c, c, a, b, c} is 3 — only three distinct objects are elements of that set. If there is no limit to the number of elements in a set, then its cardinality is ∞, and the set is **infinite**. If not, if the exhaustive roster notation for the set would eventually end, the set is **finite.**

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NAMED SETS

Some sets are so useful that they're named. Many of these you already know:

$\mathbb{Z}$ The set of integers: {…, −3, −2, −1, 0, 1, 2, 3…}
$\mathbb{R}$ The set of reals, which is essentially anything most people think of as a number
$\mathbb{Q}$ The set of rational numbers: $\{a/b \mid a, b \in \mathbb{Z}, b \neq 0\}$
$\mathbb{C}$ The set of complex numbers: $\{a + bi \mid a, b \in \mathbb{R}, i = \sqrt{-1}\}$

Some of these sets can be modified by symbols:

$\mathbb{Z}^+$, $\mathbb{R}^+$ The subsets of the integers or reals consisting only of their positive elements

One set requires care when discussing it. In the high-school level courses you've taken in math, the natural numbers were defined as {1, 2, 3, …} and given the symbol $\mathbb{N}$. The same set with zero included was called the whole numbers. Since programmers have a habit of counting from zero, the natural numbers get a different definition for this course:

$\mathbb{N}$ The set of natural numbers: {0, 1, 2, 3, …}

The set that was previously called the natural numbers are simply the positive integers, $\mathbb{Z}^+$. (We could use $\mathbb{N}^+$, but we don't see that symbol in practice.)

All the sets defined above are infinite.

There is one more named set that's very important. The **null set** or the **empty set** is the set that contains absolutely nothing in it. It's written { } or $\emptyset$. The null set is considered to be a subset of every other set. (After all, the definition of "subset" says that every element in the set must also be in the subset. This is always true — there are no elements in $\emptyset$ that aren't also in every other set.) The cardinality of $\emptyset$ is 0.

ELEMENTS VS. SUBSETS

It's a common mistake to get elements of a set and subsets of the set confused, especially in cases where sets have other sets as elements.

Example 1: Let *A* be the set comprising the sets defined in the previous section of this worksheet. Are the following objects elements of A? Are they subsets of A?
(a) 3 (b) {3} (c) $\emptyset$ (d) $\mathbb{Z}$

Solution: In roster form, set *A* is $A = \{\mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}^+, \mathbb{R}^+, \mathbb{N}, \emptyset\}$.

To decide whether something is an element of *A*, we look at it as an *object*, and determine whether that object is also in *A*.

To decide whether something is a subset of *A*, we look at it as a *set*, and compare the elements of those sets.

(a) 3 is not an element of *A*, since we can see the complete list, and 3 isn't on it. It also isn't a subset of *A*. 3 is not a set, so it can't be a subset of anything.

(b) {3} is also not an element of *A*, for the same reason as in (a). Is it a subset? 3 is an integer, and *A* contains the set of integers. The answer, however is no. **Saying {3} is a subset of *A* is the same thing as saying 3 is an element of *A***, and we already know that's not true.

(c) We know $\emptyset$ is an element of *A*, because it *is* on the list. We also know that $\emptyset$ is a

subset of A, since $\emptyset$ is a subset of every set. (There's no element in $\emptyset$ that's missing from A, so we can't say it's not a subset.)
(d) This is trickier. $\mathbb{Z}$ is an element of A, because it's on the list. Is it a subset? If $\mathbb{Z}$ is a subset of A, that's the same thing as saying the elements of $\mathbb{Z}$ are also elements of A. Since −2, −1, 0, 1, 2, 3… aren't elements of A, the answer is no, $\mathbb{Z}$ is not a subset of A.

What if A had been defined as $A = \{\mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}^+, \mathbb{R}^+, \mathbb{N}\}$? Would that change our answer to (c)? Yes! Remember, the null set is a *subset* of every set, but it is not an *element* of every set. It's only an element is it's explicitly listed or if it fits the definition of the set. So $\emptyset$ is a subset of this new version of A, but it's no longer an element.

It's important, if you're going to separate these concepts in your mind, to stop thinking of them as a "part" of the bigger set. Both an element and a subset is "part" of a bigger set, but "part" means something different in each case, so using the same term for both will only increase your confusion.

THE POWER SET

The **power set** of a set A (written "$\mathcal{P}(A)$") is a set consisting of all possible subsets of A. For the set $N = \{0, 1, 2\}$, the power set of N, $\mathcal{P}(N)$, would be $\{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$.

How do we know that we've found every possible subset? We can use a simple counting technique to find the answer.

Example 1: Let R be the set comprising the traditional colours of the rainbow. What is the cardinality of $\mathcal{P}(R)$?

Solution: We need to know how many subsets there are of the seven colours of the rainbow. We can think of the process of making a subset as a series of decisions. We have to decide first if red will be in the subset or not. We have two options there. For each of those two options we can independently decide whether orange will be in the subset or not. And so on, for the rest of the colours.

By the end of the process, we'll have made seven such decisions, and each decision can be made in two ways. Each series of decisions produces a different subset. Therefore, there are $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7 = 128$ possible subsets.

EXERCISES

A. Determine whether the two sets are equal. If not, determine whether one set is a proper subset of the other.

1) $A = \{1, 2, 3, 4, 5\}$; $B = \{3, 1, 4, 2, 5\}$
2) $C = \{q, w, e, r, t, y, u, i, o, p\}$; $D = \{t, y, p, e, w, r, i, t, e, r\}$
3) $E = \{\mathbb{R}\}$; $F = \mathbb{R}$
4) $G = \{x \mid x \in \mathbb{Z}, x > 8\}$; $H = \{9, 10, 11, 12, \ldots\}$
5) $J = \{3h + 1 \mid h \in \mathbb{Z}\}$; $K = \{3k - 2 \mid k \in \mathbb{Z}\}$
6) $L = \{a/b \mid a, b \in \mathbb{Q}, b \neq 0\}$; $M = \mathbb{Q}$
7) $N = \{p^2 \mid p \in \mathbb{R}\}$; $P = \mathbb{R}^+$
8) $Q = \{ab \mid a, b \in \mathbb{Z}\}$; $R = \{c + d \mid c, d \in \mathbb{Z}\}$

B. Determine the cardinality of the 8 sets in Question A (1)–(4).

C. Each of the following questions defines a set, and then lists a number of objects. Determine which of those objects are elements of the set.

1) $A = \{x \mid (x \in \mathbb{R}) \wedge (x \notin \mathbb{Q})\}$
   $\ln 2$, π, 0, $\emptyset$, $6/0$, $\overline{\mathbb{Q}}$

2) $B = \{x \mid x = 3a + b;\ a, b \in \mathbb{Z}^+\}$
   15, 3, 6, −4, 7.5, 574

3) $C = \{\mathbb{C}, \mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}\}$
   $\mathbb{R}^+$, the imaginary numbers, $\emptyset$

4) $D = \{1, 2, 3, \{1, 2, 3\}\}$
   {3}, 2, {2, 1, 3}, {1, 1}, {3, 1, 2, 3}

5) $E = \mathcal{P}(C)$, where C is the set from question (3): $\mathbb{Q}$, $\{\mathbb{N}, \mathbb{R}\}$, $\emptyset$, $\{\emptyset\}$, C

6) F = {yes, no, {maybe, yes}, {maybe, no}}
   no, maybe, {yes, no}, {yes, maybe}

7) G = {6, {6}, {6, 7}, {6, {6}}, {7, 7}}
   6, 7, {6}, {7}, {{6}}

8) $H = \mathcal{P}(G)$, where G is the set from question (7): {6, {6, {6}}}, {6, 6}, {{7}}

D. Each of the following questions defines a set, and then suggests a number of subsets. Determine which of the suggestions actually is a subset of the first set.

1) K = {January, June, July, December}
   {June, July}, December, $\{x \mid x$ is a month of the year$\}$, $\emptyset$

2) $L = \{1, 3, 5, 7, 9, \ldots\}$
   {5, 7, 5}, {1221}, $\{2x + 5 \mid x \in \mathbb{N}\}$, $\{p^2 \mid p \in L\}$, ∞

3) M = {a, b, c, {a, b}, {a, b, c}}
   {{a}}, {a, b}, {a, c}, {{a, b}}, {{a, c}}

4) N = {zero, the letter O, the null set}
   0, $\emptyset$, $\{\emptyset\}$, $\{0, \text{O}, \emptyset\}$, the power set of the set {0}

5) $P = \{1, \{1\}, \{\{1\}\}, \{\{\{1\}\}\}, \ldots\}$
   {{1}}, {1, {1}}, {{1}, {1}}, $\{1, \emptyset\}$, {{{{{1}}}}, {{{{{{1}}}}}}}, {{{{1}, {{1}}}, {{{{1}}}}}

6) $Q = \{x \mid x$ is a non-zero integer$\}$
   $\mathbb{N}$, $\mathbb{Z}^+$, $\{\lfloor a/b \rfloor \mid a, b \in \mathbb{Z}^+\}$, $\{s \mid s = |\mathcal{P}(S)|$ for some finite set $S\}$, $\{a + b \mid a, b \in Q\}$

SOLUTIONS

A: (1) $A = B$ (2) $D \subset C$: q $\notin D$ (3) not equal; neither is a subset of the other: the one element in E is a set, not a real number
(4) $G = H$ (5) $J = K$: the formulas produce the same elements when $h = k - 1$
(6) $L = M$: the quotient of two rational numbers is a rational number, and every element m in M is represented in L when $a = m$ and $b = 1$
(7) $P \subset N$: $0 \in N$ but $0 \notin P$
(8) $Q = R$: for any integer z, $z \in Q$ when $a = z$ and $b = 1$ and $z \in R$ when $c = z$ and $d = 0$

B: (1) $|A| = |B| = 5$ (2) $|C| = 10$, $|D| = 7$ (3) $|E| = 1$, $|F| = \infty$ (4) $|G| = |H| = \infty$

C: (1) $\ln 2$, π (2) 15, 6, 574 (3) none of them (4) 2, {2, 1, 3}, {3, 1, 2, 3}
(5) $\{\mathbb{N}, \mathbb{R}\}$, $\emptyset$, C (6) no, {yes, maybe} (7) 6, {6}, {7} (8) all of them

D: (1) {June, July}, $\emptyset$ (2) {5, 7, 5}, {1221}, $\{2x + 5 \mid x \in \mathbb{N}\}$, $\{p^2 \mid p \in L\}$
(3) {a, b}, {a, c}, {{a, b}} (4) $\emptyset$, $\{\emptyset\}$, $\{0, \text{O}, \emptyset\}$
(5) {{1}}, {1, {1}}, {{1}, {1}}, {{{{{1}}}}, {{{{{{1}}}}}}}
(6) $\mathbb{Z}^+$, $\{s \mid s = |\mathcal{P}(S)|$ for some finite set $S\}$, $\{a + b \mid a, b \in Q\}$