Open Statements & Quantified Statements

Closed statements are statements about specific things that are absolutely true or false. A statement like “February 14 is Valentine’s Day,” is always true.

What about the argument: “Every day has 24 hours. Therefore Valentine’s Day has 24 hours.” This argument uses more complex relationships than we’ve seen. It’s clearly a valid argument, but it’s awkward to express using closed statements. It uses a category, and applies the properties of the category to a specific thing within the category. This idea of a property that an object has is best expressed as an **open statement**.

An open statement is written as $P(x)$, and it tells us that object x has property P. For example, $P(x)$ might be defined as “$P(x)$: x has 24 hours.” These statements can have multiple inputs: $Q(x, y)$: x is bigger than y.

We introduce a symbol that means that a statement applies to every object: $\forall x$ (for all x). We could write the sentence “Every day has 24 hours,” as $\forall x\ P(x)$. We need to worry about the scope of “every”, however. Do we mean literally everything? Cats have 24 hours, the Eiffel Tower has 24 hours, sincerity has 24 hours…. We will also need to state what sort of object the statement describes. We call this the **universe**. Our argument is valid over the universe of days, which contains Valentine’s Day:

Let U be the universe of days. Let $P(x)$ = “x has 24 hours”. Let $v \in U$. Then:

$\forall x\ P(x)$
$\therefore\ P(v)$

This argument is valid because of the **Law of Universal Specification**: If an open statement applies to every object in the universe, then we can specify an object from the universe, and the open statement applies to it.

The choice of universe is important. Consider the argument, “Every day has 24 hours. February doesn’t have just 24 hours. Therefore February is not a day.” This looks like we can create this argument in the same framework as the previous argument, except for our conclusion. If we wish to prove that something is not a day, that’s essentially saying it’s not part of the universe. That’s a problem. We’ll need to change our definition of the universe to include February. We can create a second open statement for the property of being in the category of days:

Let U be the universe of objects.
Let $P(x)$ = “x has 24 hours”. Let $D(x)$ = “x is a day”. Let $f \in U$. Then:

①	$\forall x\ [D(x) \to P(x)]$	Given	For all x, if x is a day, then it has 24 hours.
②	$\neg P(f)$	Given	February does not have 24 hours.
③	$D(f) \to P(f)$	Univ. Spec. ①	If February is a day, then it has 24 hours.
④	$\therefore \neg D(f)$	Mod. Toll., ③, ②	Therefore, February is not a day.

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We took a **universally quantified** statement (a statement that talks about everything in the universe) and used it to make a statement about a specific thing in the universe. Once we did that, all of the propositions in lines 2–4 behave like closed statements. (After all, the difference between open and closed statements is whether they apply to a class of objects or a specific object.)

We can reverse the process, using information about specific objects in the universe to make broader statements, by referring to a single generic member of the universe, proving something about it, and expanding the statements to describe the universe, by the **Law of Universal Generalization** (a powerful method of proof which you'll see in use later in the course), or we can find an example which proves a universally quantified statement is not true. Such an example is called a **counterexample**.

Consider this argument: "2000 is a year. 2000 did not have 365 days. Therefore not every year has 365 days." The conclusion of this argument negates a universally quantified statement. What does it mean to do that? It's not the same thing as "all years do not have 365 days," — that's just as false as saying "all years have 365 days." The negation means that *some* years don't have 365 days, but not necessarily all years. We're now discussing the existence of objects with certain properties within the universe. We use a second quantifier, $\exists x$, ("there exists x") when we want to say that at least one object in the universe satisfies certain conditions:

$$\neg\forall x\, P(x) \Leftrightarrow \exists x\,[\neg P(x)]$$

What happens when we write $\neg\exists x\, P(x)$? If it's not true that an object with a particular property exists, then it's true that every object doesn't have that property:

$$\neg\exists x\, P(x) \Leftrightarrow \forall x\,[\neg P(x)]$$

The **Law of Existential Generalization** says that if we can describe an object in the universe with a property, then we can say that there exists an object in the universe with that property. Now we can construct the argument about years:

Let U be the universe of objects.
Let $A(x)$ = "x is a year". Let $D(x)$ = "x has 365 days". Let $t \in U$. Then:

①	$A(t)$	Given	2000 is a year.
②	$\neg D(t)$	Given	2000 did not have 365 days.
③	$\neg[\neg A(t)] \wedge \neg D(t)$	Conj. ①, ②, D.Neg.	
④	$\neg[\neg A(t) \vee D(t)]$	DeMorgan's ③	
⑤	$\neg[A(t) \rightarrow D(t)]$	Implication ④	
⑥	$\exists x\, \neg[A(x) \rightarrow D(x)]$	Exist.Gen., ⑤	There is a year that does not have 365 days.
⑦	$\therefore \neg\forall x\, [A(x) \rightarrow D(x)]$	Mod. Toll., ③, ②	Not every year has 365 days.

We've seen laws of Universal Generalization, Universal Specification, and Existential Generalization. There *is* a **law of Existential Specification**: if we can say that there exists an object with certain properties then we may discuss such an object.

When dealing with more than one object, there may be more than one quantifier. To express the idea "The sum of any two integers is also an integer," could be written: $\forall x, y\, [[(x \in \mathbb{Z}) \wedge (y \in \mathbb{Z})] \rightarrow (x + y \in \mathbb{Z})]$.

They can also be mixed, one universal, one existential. The phrase "There's someone for everyone," means "For everyone, there is a suitable partner." As a quantified statement (with the obvious definitions), we would write this as $\forall x\ \exists y\ [P(x, y)]$. When you have both kinds of quantifiers, the order matters! The same statement with the quantifiers reversed, $\exists y\ \forall x\ [P(x, y)]$, would mean "There is a person who could be a suitable partner for everyone." That's very different!

Negations work on multiple quantifiers in the obvious way. The negation of the statement, "There is a person who could be a suitable partner for everyone," would go like this:

$\neg[\exists y\ \forall x\ [P(x, y)]]$
$\Leftrightarrow \forall y\ \exists x\ [\neg P(x, y)]$ For every person, there is someone who would not be a suitable partner.

EXERCISES

A. Let U be the universe of objects. Use the definitions provided to interpret the quantified statements as familiar English expressions, or vice versa.

$A(x)$: "x is gold"	$H(x)$: "x is happiness"	$Q(x)$: "x is water"
$B(x)$: "x is bad"	$J(x)$: "x is a stage"	$R(x)$: "x rolls"
$C(x, y)$: "x collects y"	$K(x, y)$: "x is better than y"	$S(x)$: "x is blood"
$D(x)$: "x is money"	$L(x)$: "x is a tree"	$T(x, y)$: "x is thinner than y"
$E(x, y)$: "x buys y"	$M(x)$: "x is moss"	$W(x)$: "x is in the world"
$F(x)$: "x is free"	$N(x)$: "x is an apple"	$Y(x)$: "x is correct"
$G(x)$: "x glitters"	$P(x)$: "x is a stone"	$Z(x, y)$: "x is a source of y"

1) $\forall x\ [W(x) \rightarrow J(x)]$
2) Money isn't everything.
3) $\forall x, y\ [[D(x) \wedge L(y)] \rightarrow \neg Z(y, x)]$
4) You can't get blood from a stone.
5) $\forall x, y\ [E(x, y) \rightarrow Y(x)]$
6) All that glitters is not gold. (Not everything that glitters is gold.)
7) $\forall x, y\ [[P(x) \wedge R(x) \wedge M(y)] \rightarrow \neg C(x, y)]$
8) Money can't buy happiness.
9) $[\exists x\ [N(x) \wedge B(x)]] \rightarrow [\forall x\ [N(x) \rightarrow B(x)]]$

B. Negate the following statements. (You don't need to do anything to the information before the colon ":".)

1) $\forall x\ [P(x) \vee Q(x)]$
2) $\exists x\ [P(x) \rightarrow Q(x)]$
3) $\forall x,y\ [[P(x) \vee Q(y)] \rightarrow S(x, y)]$
4) With $x, y \in \mathbb{R}$: $\forall x\ \exists y\ [x \cdot y = 1]$
5) With $x, y, z \in \mathbb{R}$, $x < y$: $\forall x, y\ [[f \text{ is a continuous function}] \wedge [|f(x)|/f(x) \neq |f(y)|/f(y)]] \rightarrow \exists z\ [[z \in (x, y)] \wedge [f(z) = 0]]$

C. Which law (Univ. Spec., Univ. Gen., Exist. Spec., Exist. Gen.) justifies the following underlined statements?

1) To prove: If n is an even number, then n^2 is an even number.
   Let n be an even number. <u>Then we may write n = 2m for some integer m.</u> …

2) …Thus $n^2 = 4m^2 = 2(2m^2)$. We have said m is an integer, so <u>m^2 is an integer</u> and so is $2m^2$, since the product of two integers is an integer. …

3) … We have written n^2 as 2 times an integer, so n^2 is an even number. <u>Therefore the square of an even number is an even number.</u>

4) To prove: The set of integers has only one identity element under multiplication.
   We will first prove that an identity element exists. For any integer, n, $1 \times n = n \times 1 = n$. Since 1 satisfies the definition of an identity element, <u>a multiplicative identity exists.</u>

5) … We will now prove that there aren't multiple multiplicative identities by contradiction. Assume there is more than one distinct identity element. <u>Consider two identity elements, a, b $\in \mathbb{Z}$</u>, $a \neq b$.

D. Determine whether the following arguments are valid (by proving the conclusion) or invalid (by proving the negation of the conclusion or otherwise finding a counterexample).

1) Bunnies have long ears. Peter has long ears. **Therefore** Peter is a bunny.

2) Some vehicles are streetcars. If a vehicle is not a streetcar, it's a hovercraft. Desire is a vehicle, but not a hovercraft. **Therefore** Desire is a streetcar.

3) Seedless grapes need to be grafted to proliferate. Neptune grapes are seedless. **Therefore** some grapes need to be grafted to proliferate.

SOLUTIONS

A: (1) All the world's a stage. (2) $\exists x\,[\neg D(x)]$ (3) Money doesn't grow on trees.
(4) $\forall x, y\,[[P(x) \wedge S(y)] \to \neg Z(x, y)]$ (5) The customer is always right.
(6) $\exists x\,[G(x) \wedge \neg A(x)]$ (7) A rolling stone gathers no moss.
(8) $\forall x, y\,[[D(x) \wedge H(y)] \to \neg E(x, y)]$ (9) One bad apple spoils the bunch.

B: (1) $\exists x\,[\neg P(x) \wedge \neg Q(x)]$ (2) $\forall x\,[P(x) \wedge \neg Q(x)]$ (3) $\exists x, y\,[[P(x) \vee Q(y)] \wedge \neg S(x, y)]$
(4) $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ [x \cdot y \neq 1]$
(5) $\exists x, y\,[[f \text{ is continuous}] \wedge [|f(x)|/f(x) \neq |f(y)|/f(y)]] \wedge \forall z\,[[z \notin (x, y)] \vee [f(z) \neq 0]]$

C: (1) Universal Specification (2) Universal Specification
(3) Universal Generalization (4) Existential Generalization
(5) Existential Specification

D: In these solutions, open statements and objects are represented by obvious initials.
(1) Invalid: other animals could have long ears (e.g., donkeys) and Peter could be one of those.
(2) Valid: Let U be the universe of vehicles. ① $\exists x\, S(x)$ {Given} ② $\forall x\,[\neg S(x) \to H(x)]$ {Given} ③ $\neg H(d)$ {Given} ④ $\neg S(d) \to H(d)$ {Univ.Spec. ②} ⑤ $S(d)$ {Mod.Toll. ④, ③} ■
(3) Valid: Let U be the universe of grapes. ① $\forall x\,[S(x) \to G(x)]$ {Given} ② $S(n)$ {Given} ③ $S(n) \to G(n)$ {Univ.Spec. ①} ④ $G(n)$ {Mod.Pon. ③, ②} ⑤ $\exists x\, G(x)$ {Univ.Gen. ④} ■