Essentials of Linear Algebra

It can feel like you're doing the same thing over and over again in the linear algebra course. There are a lot of parallels to be drawn between types of problems, but it can be hard to straighten everything out to see those parallels.

There are three "worlds" that linear algebra problems in this course inhabit. We'll take a look at the simplest possible problem in each one, and hopefully that will build a framework that lets us see how all three are really talking about the same thing. In all cases, we'll look at a "three-dimensional" problem, since the bulk of the work in the course is based on that.

SYSTEMS OF EQUATIONS

We've seen systems of equations before: the goal is to find the solution or solutions that satisfy all the equations at once. (An older name for this kind of problem is a system of *simultaneous* equations.) The simplest possible problem in "three dimensions" looks like this:

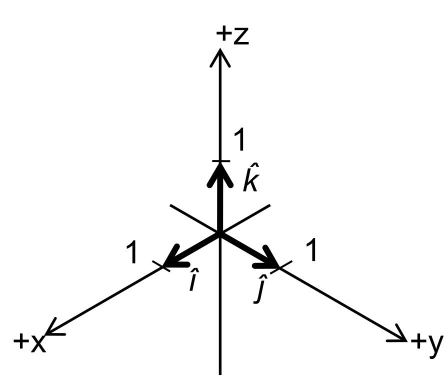
$$\begin{aligned} u_1 \qquad\qquad &= b_1 \\ u_2 \qquad &= b_2 \\ u_3 &= b_3 \end{aligned}$$

That's pretty simple. The variables are on the left, and the constants that those variables take on are on the right.

VECTORS IN SPACE

We've seen vectors in physics before as well: arrows that we can add or take scalar multiples of, and which can represent forces, velocities, and other things. They have magnitude and direction (how far, and which way), but not position (where). This means that we are free to move vectors to a new position if it helps us with a calculation.

In linear algebra, we're chiefly concerned with using vectors to get from origin to a generic point in space. The simplest possible solution of what vectors allow us to reference any point in space is the three unit vectors in the directions of the three coordinate axes. If we needed to combine these vectors to get the resultant $\langle b_1, b_2, b_3 \rangle$ we know we can scalar-multiply each one by b_i and add the three vectors together.



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MATRICES

We also have a new class of problem, dealing with matrices. Specifically, we're looking at the problem $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The simplest version of this problem is when the **coefficient matrix**, *A*, happens to be the **identity matrix**, *I*:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Then $x = b_1$, $y = b_2$ and $z = b_3$.

DRAWING PARALLELS

It's not hard to see what the matrix problem and the system of equations problem have to do with each other: if we perform the matrix multiplication in the standard matrix problem, we get a column vector whose elements are a generic three-dimensional system of equations:

$$\begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \\ a_{31}x + a_{32}y + a_{33}z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

So the matrix equation is a representation of a system of equations. What about the vector problem? You probably noticed that the variable b was used in all three examples. We will frequently write a three dimensional vector as a matrix — that's why a matrix with either one row or one column is called a vector — and we'll also frequently view a matrix with at least two columns and at least two rows as a series of column vectors fused together. The identity matrix is just the basic unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ fused together into one matrix. In this case you can view the system of equations problem as finding scalar multiples of several vectors that add up to a target vector's x-, y- and z-components: Given three vectors, $\langle a_{11}, a_{21}, a_{31}\rangle$, $\langle a_{12}, a_{22}, a_{32}\rangle$, and $\langle a_{13}, a_{23}, a_{33}\rangle$, is it possible to express $\langle b_1, b_2, b_3\rangle$ as a linear combination of them:

$$\left[\begin{array}{c:c:c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$$u_1\langle a_{11}, a_{21}, a_{31}\rangle + u_2\langle a_{12}, a_{22}, a_{32}\rangle + u_3\langle a_{13}, a_{23}, a_{33}\rangle = \langle b_1, b_2, b_3\rangle$$

The only way to get the vectors to be equal is for the x-, y- and z-components are equal:

$$\begin{aligned} u_1a_{11} + u_2a_{12} + u_3a_{13} &= b_1 \\ u_1a_{21} + u_2a_{22} + u_3a_{23} &= b_2 \\ u_1a_{31} + u_2a_{32} + u_3a_{33} &= b_3 \end{aligned}$$

This is the system of equations again, which means it can be represented by a matrix built out of column vectors. So the matrix problem has two major classes of interpretations/applications: systems of equations and vectors.

TOO MUCH OF THIS, TOO LITTLE OF THAT

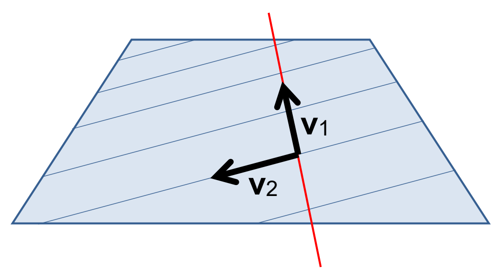
Each of the simplest problems we've seen is "balanced". We know from our work with systems before this course that we like it when there are exactly as many equations as there are variables to solve for—three equations, three unknowns—because then we can expect to be able to solve the problem, and it likely has exactly one numerical answer. One of the themes explored in linear algebra is, what if there is an imbalance?

If a system of equations has fewer equations than it has unknowns, then there is not enough information to solve the problem. Thus the system is more likely to be **consistent** (more likely that there is a solution, possibly many solutions) and less likely to be **dependent** (less likely that the system has a redundant equation). If it has more equations than it has unknowns, it's more likely that the system is **inconsistent** (more likely that the information is contradictory, and thus there is no solution), and it means the system *must* be dependent. If three equations is sufficient to pin down one solution in the best case, then a fourth equation can't tell us anything new. The system must have a redundant equation in it.

What about the vector problem? It's easier to understand if we visualize the results of all possible linear combinations of a given set of vectors. We'll assume that the tail end of all these vectors will always be on the origin of coordinates.

If we start with only one vector, $\mathbf{v}_1 = \langle a_{11}, a_{21}, a_{31} \rangle$, then we can only take scalar multiples of that vector, and those resultants will all have the same direction as $\mathbf{v}_1$, so all the resultants lie on the same straight line.

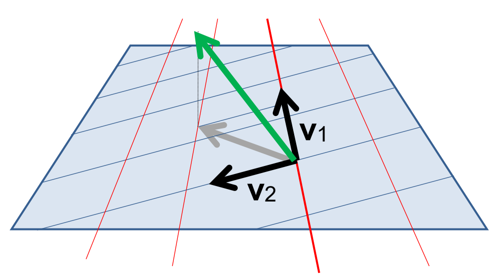
If we then add a second vector which is not parallel to the first, $\mathbf{v}_2 = \langle a_{12}, a_{22}, a_{32} \rangle$, then linear combinations of those two vectors let us move anywhere on a flat surface, a plane. In a sense, the two vectors *can* act like $\hat{\imath}$ and $\hat{\jmath}$, forming a less-convenient set of axes for the plane.



The vector problem wants to know whether a linear combination of a set of vectors can equal a target vector. If we're only given two vectors for three-dimensional space, then it's very possible that the target vector lies outside the plane defined by the two given vectors; it's more likely that the target is **inconsistent** with the set — no possible solution.

This feels like it should be the analogue of a system with too few equations. It's not! It's a two-dimensional problem, since we're trying to see whether the target lies on a particular *plane*. The plane is defined with three components, and therefore three equations:

$$\begin{aligned} u_1 a_{11} + u_2 a_{12} &= b_1 \\ u_1 a_{21} + u_2 a_{22} &= b_2 \\ u_1 a_{31} + u_2 a_{32} &= b_3 \end{aligned}$$



It's actually the too-many-equations situation. Two equations might work, while the third one doesn't.

Adding a $\mathbf{v}_3 = \langle a_{13}, a_{23}, a_{33} \rangle$, so long as it doesn't lie in the same plane as the other two, gives you a way to get anywhere in the xyz-space. Since each vector allows movement in a direction that the others can't provide, all three are essential, and there's no redundancy.

So what if we add a fourth vector? Since the original three from our example already cover all of three-dimensional space, the fourth vector doesn't bring anything new to the table. What if the first three didn't span all of $\mathbb{R}^3$? Well, then there's already some redundancy, and the set of vectors is dependent. No matter what, four (or more) vectors in three-dimensional space must have a dependency somewhere. We can say that *n* independent vectors with *n* components will span $\mathbb{R}^n$; if there are more than *n*, then the set of vectors must be dependent.

HOMOGENEOUS SYSTEMS

Reducing a problem to a simpler problem can help us understand it better. That's why we study homogeneous systems. If we're exploring what sorts of constants will or will not have a unique solution (or any solution) in a system of equations, then there's something to be learned from the case where the constants are all 0.

Solving this system is usually much easier than solving one with non-zero elements in **b**. If the generic, non-homogeneous system has only one solution, then the homogeneous one also has only one solution, and that solution is always the all-zero **trivial solution**. The homogeneous system is also never inconsistent; the trivial solution is *always* valid. It does however let us tell the difference between a dependent system and an independent system: if the homogeneous system has solutions other than the trivial one, so will a related consistent, nonhomogeneous system. (In this case, the nonhomogeneous system may not have a solution at all, which is why we have the word "consistent" in the conclusion.)

The geometric interpretation of using a homogeneous system is this: if the nonhomogeneous system has one solution — i.e., the intersection of the objects each equation represents is one point in space — the homogeneous system translates that point to the origin. If the solution space of the nonhomogeneous system is infinite (it's a line, plane, …) then the solution space of the homogeneous system has the same dimension (it's also a line, or also a plane, …) and it's "parallel" to the nonhomogeneous system's solution space.

THEREFORE, MATRICES

Since all these applications and interpretations of these problems overlap each other so heavily, we investigate them through numerical matrices and vectors, since mathematicians are very good at picking apart numerical systems. All the other things you'll learn about in linear algebra will be about the algebraic properties of matrices (how to do calculations with them, what properties they have, how you can analyze a matrix to determine the nature of a solution space without having to do the whole calculation, …) and then what the consequences of those properties are for systems of equations and geometry.