Fundamentals of Graphs

Graphs are powerful. They let us see all the solutions to a formula at once. They simplify questions about domain and range, and they illustrate the behaviours of functions that are difficult to see from just the formula. They're a useful tool for extracting additional information about trends or other ordered pairs that satisfy the formula. This worksheet talks about the basics that help us create and understand graphs.

Variables have a few different jobs in algebra. Sometimes they refer to specific numbers whose identity we don't know, although we do know how the number behaves in some equation or equations. If you see the equation:

$$x + 3 = 5$$

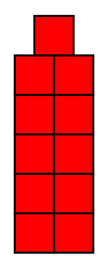
…then you can see what single number is the only value for x that will make the equation true. Here, we're *solving* for a variable. We haven't been told the value, but we've been told how in behaves in an expression.

Another job that variables do is that they're markers for all possible numbers at the same time. Just like pronouns in English, they help us keep track of numbers and their relationships without needing to get specific. We use this when we're trying to demonstrate that a particular mathematical statement is always true, no matter what numbers are used. A statement like this:

$$3x(x + 5) = 3x^2 + 15x$$

…you should recognize as just multiplication. It's true for every number that you might plug in for x; that's *why* we multiply that way. It always works.

Aside from these statements, we use variables to represent all numbers in **formulas**, sets of instructions that tell you how to calculate things.



Example 1: A child builds towers of different heights, with two blocks in each level and one extra block on top. The example shown at the right is a five-level tower (not six!). How many blocks will the child need to build a tower of a particular height?

Solution: The tower we see has 11 blocks, but that's not true for towers that are taller or shorter. No single number is an answer to this question, since the answer depends on the height of the tower. The answer should be a formula.

In order to answer this question in a meaningful way, we have to assume that we already know how tall we want the tower to be. Once we know that, there's one correct answer to the question, "How many blocks do I need?" (This is how formulas work in general: one number determines another number.)

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If we examine the example, we can see how to figure out the answer: We'll need two blocks for every story in the tower, plus one more to go on the top. If we want to build a tower with 7 stories, we'll need $2 \times 7 = 14$ blocks, plus 1 for the top, which is 15. For 10 stories, $2 \times 10 + 1 = 21$. For 2 stories, $2 \times 2 + 1 = 5$. And of course, for 5 stories, $2 \times 5 + 1 = 11$. The numbers may change, but the *process* is the same every time. Formulas describe a process.

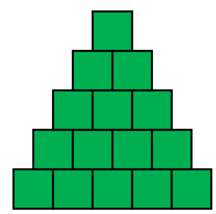
To write the formula for the number of blocks needed, we pick a variable for the number we know: the number of stories. X is usually used for the number we know. Then we write down what calculations we do to x to find the number want to know (how many blocks). For the building block tower, that's $2x + 1$ — take The Number of stories, multiply it by 2, and then add 1. The number we find out by doing a calculation with x is usually called y:

$$y = 2x + 1$$

Some notes:

- Formulas may have more than one input number or *argument*. For graphing, you'll only work with formulas that have one variable (input number that changes) — any other input is written as a number that doesn't change, known as a **constant** (like the 2 in the building block tower formula).

- Formulas may use a variable more than once. If each row of the tower increased in size, like the one on the right here, the formula would be $y = ½(x^2 + x)$. The formula mentions *x* twice, so that means that *x* is the same number every time in this formula. This tower is five levels tall, and the number of blocks, $y = ½(5^2 + 5) = ½(30) = 15$.



- In a word problem, variables may have restrictions on what kind of number *x* can be. For the tower problem, *x* needs to be a natural number: that is, no fractions, no negative numbers, and no zero. On the other hand, as an abstract formula, there's nothing stopping you from plugging −3.2 into the formula and getting a number out. It's simply the case that this answer has no meaning for the original problem. A child can't use −5.4 blocks to build a tower. For graphing, these restrictions won't be in place, and any real number might be substituted for *x*.

AT LONG LAST, GRAPHING

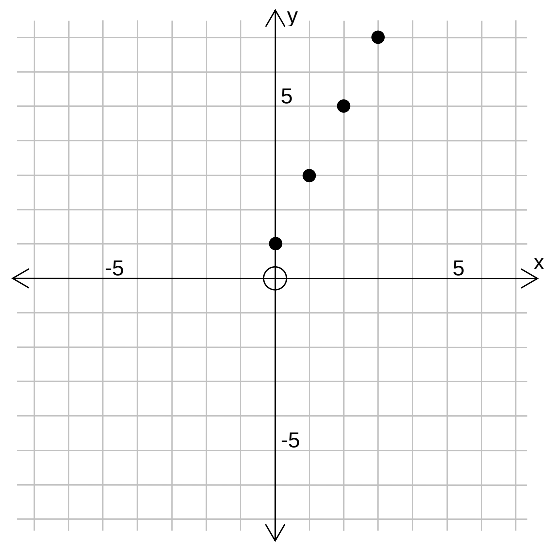
If you've ever played with Google Earth or looked in an atlas, you may have noticed that every location on Earth has its own set of **coordinates** to describe exactly where it is. The Broadway campus of Vancouver Community College is at 49° 15' 45.14" N 123° 4' 50.03" W. Those two numbers, one going east-west and one going north-south, are enough to specify the location.

The formula that we got from the building block tower turns *x* values into *y* values, and these can be interpreted as coordinates. The instructions in the formula form a pattern, and drawing the results of the formula creates a picture that helps us understand the problem. The picture is a **graph**.

There are some standards that we use in creating graphs. The horizontal axis is called the **x-axis**, and it behaves just like the standard number line: positive numbers getting bigger to the right, and negative numbers to the left. We draw a vertical axis so that it crosses the x-axis at 0. This is the **y-axis**. Remember that the value of *x* determines the value of *y*. Even if we don't use "*x*" and "*y*" as our variables (we rarely do when we're making a graph for a science lab, for example) the relationship holds true: the number we use from the horizontal axis (the **independent variable**) determines the value for the vertical axis (the **dependent variable**). We write these coordinates as an **ordered pair** — in parentheses, separated by a comma, as in (5, 11). *x* comes first, and that gives you *y*; 5 stories gives you 11 blocks.

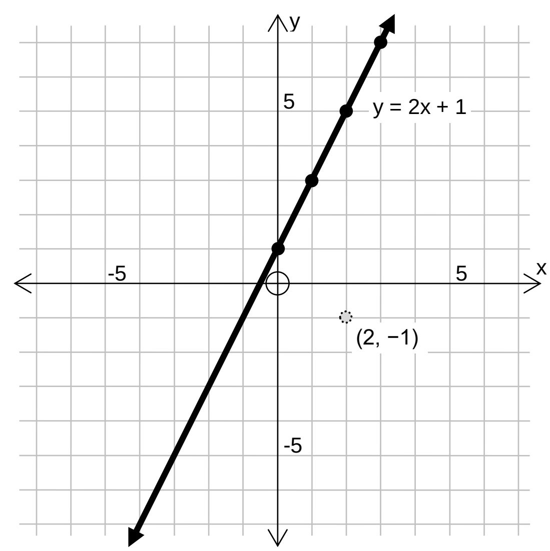
When we interpret a formula as coordinates, we can create a **table of values** to track all the numbers. We draw a dot for the pairs of numbers that the formula generates, and we see a pattern: the dots all lie on a straight line. We can use the pattern to predict other values. (If this was experimental data, we would just have (x, y) pairs and no formula. It's easier to predict a value from the graph than from a list of numbers.)

x	*y*
0	1
1	3
2	5
3	7
4	9
5	11



Notice that some of the points, such as (4, 9) and (5, 11) don't fit on this graph. They're still part of the formula. You can assume that the graph paper will continue on forever in all directions, and any lines or curves will also continue in a predictable way.

If we treat y = 2x + 1 as a formula where *x* can be any real number, then we could use these points to confidently complete the graph by drawing the straight line that passes through all these points. We put arrowheads on both ends of the line to indicate that it continues forever in both directions.



The interpretation of a graph like this is that all the (x, y) pairs of numbers that satisfy the formula y = 2x + 1 are on the line, and any pairs of numbers that aren't on the line don't satisfy the equation. The point (2, −1) doesn't work for the equation; an *x*-value of 2 gives you a *y*-value of 5, not −1. The point (2, −1) is marked on this graph, and it's nowhere near the line. Completing the graph gives us a picture of all solutions at once; something you could never do by filling the table of values — think of all the weird decimal numbers you'd have to try! There are infinitely many of those, so we'd never finish.